Law of Small Numbers in a Lucas Economy

Abstract

Tversky and Kahneman (1971) document that agents follow the Law of Small Numbers when updating beliefs, that is, they believe that small samples should be representative of the population. We propose a continuous time model of the belief in the Law of Small Number and study the asset pricing and portfolio choice implications of this bias within an N-trees Lucas economy with agents having heterogeneous beliefs. The following empirical irregularities arise naturally within the economy we consider: (1) A subset of agents exhibit higher V-shaped trading patterns, that is, they have a (lower) likelihood to sell (buy) winners relative to losers; (2) The disposition effect; (3) Momentum; (4) A high average (and time varying) equity premium; (5) Excess volatility; (6) Price underraction to dividend news.

Keywords: Lucas Economy, Law of Small Numbers, Heterogeneous Beliefs
A standard assumption in Finance and Economics is that agents use Bayes’ rule when updating their beliefs. However, Tversky and Kahneman (1971) document that agents tend to depart from complete rationality when updating beliefs and instead follow that they call the Law of Small Numbers (LSN). The belief in the LSN is the mistaken belief that a small sample drawn from a population should be representative of that population. We study the portfolio choice and asset pricing implications of the belief in the LSN. Beliefs about an asset’s future payoff/dividend stream have a direct effect on agents’ individual demand for the asset. This effect depends on how the beliefs influence both expected future cash-flows and expected future discount rate. In the absence of frictions, the effect of beliefs on prices will depend on the both the beliefs dynamics and dispersion.

We consider a continuous time (endowment) Lucas economy with two added features. First, we allow for belief heterogeneity. A subset of agents, who we collectively refer to as Freddy, believes in the LSN when forecasting future dividend growth while the remaining agents follow the rational assumption. In equilibrium, shocks to beliefs impact both expected future cash-flows and expected future discount rates within endowment economies like the one we model. The second feature is the presence of multiple trees which allows the interaction between beliefs and cash-flow news from that between beliefs and discount rate news.

The belief in the LSN has implications for updating beliefs because it generates the Gambler’s fallacy, that is, the belief in reversal. For illustration, suppose that Freddy observes sample $S_n$ of $n$ independent coin tosses from a fair coin. Moreover, assume that the number of heads is higher than the number of tails in $S_n$. Then, belief in the LSN...
implies that Freddy believes the next coin toss has a distribution with the property that tail occurs with probability higher than 50%. That is, Freddy’s belief about the distribution of the \((n + 1)\)th coin toss is such that the expected value of the random sample \(S_{n+1}\) is closer to the population mean relative to the mean of \(S_n\).

We propose a continuous time model of belief in the LSN that can be used within a continuous time Lucas economy. Freddy’s subjective expected dividend growth differs from the true expected dividend growth by a term that is the negative of a weighted sum of past shocks to dividend growth, where recent shocks have higher weights. The intuition is that shocks to dividend growth cause the realized dividend growth to be different from its expected value. The weighted sum an aggregation of the deviations from the true expected dividend growth. Freddy subtracts this sum from the true dividend growth when forming expectation because he believes that the flow of next deviations will cancel out the past deviations. The model yields the Gambler’s fallacy: Freddy’s subjective expected dividend growth is higher (lower) after observing a recent history of shocks to dividend growth whose aggregate value is negative (positive). We refer to such history as a recent negative (positive) history.

Belief in the Law of Small Numbers leads Freddy to exhibit an asymmetric V-shaped trading patterns within the heterogeneous Lucas economy we consider: Both increases and decreases in returns are associated with non-zero probabilities that Freddy will sell the risky asset. However, the likelihood of selling the risky asset is higher for an increase in returns relative to a decrease in returns. The opposite pattern holds for buys: Freddy buys the risky asset more frequently following a decrease in returns relative to an increase in returns. These asymmetric V-shaped patterns are consistent with the empirical findings of Ben-David and Hirshleifer (2012).

Freddy exhibits the disposition effect in our model. Following Odean (1998), we com-
pute both Freddy’s percentage of realized gains (PRG) and Freddy’s percentage of realized losses every months using simulated data and find that the PRG is greater than the PRL. The disposition effect follows from the V-shaped sell pattern. Conditional on selling, Freddy sells a higher proportion of risky assets whose returns have gone up relative to those whose returns have gone down. In addition, Freddy’s average percentage change in holding of the risky asset is higher for the risky stocks with the lowest returns relative to those with higher returns. Thus, on average, Freddy decreases (increases) her holding of the risky asset following an increase (decrease) in returns.

An asset pricing implication of the LSN is a higher equity premium relative to the benchmark case without Freddy. Freddy’s belief induces additional risk for the rational investor because there is a non-zero probability that future dividend growth will be more in line with Freddy’s expectations. This risk reduces overall the demand of the risky asset and results in lower prices and higher returns. The equity premium is increasing in the magnitude of disagreement between the two agents about the mean dividend growth.

The time-varying beliefs also generates high trading volume and excess volatility. Agents trade because they disagree about the expected dividend growth. Freddy changes her belief frequently which causes greater fluctuations in prices. We find that excess volatility tends to be higher (lower) when Freddy is moderately optimistic (pessimistic). Volatility is lower when Freddy is either very pessimistic or very optimistic because one of the two agents is reluctant to hold the risky asset in such case. As a result, the risk-free rate is higher in this case.

Momentum is present in our model. That is, a portfolio of past winners outperforms a portfolio of past losers in the short run. (Recent) Past winners are associated with high realized dividends. These high realized dividends lead to lower prices, and thus high future returns, because Freddy expects low future dividends. The momentum effect is stronger
when Freddy holds a higher share of consumption, consistent with the effect been driven
by the belief in the LSN. The same mechanism leads to positive autocorrelation in returns
for individual stocks, consistent with time-series momentum (see Moskowitz et al. (2012)),
and to price underreacting to dividend news.

The novel element in our model is the belief in the LSN. Tversky and Kahneman
(1971) document this belief within a group of mathematical psychologists. Rapoport and
Budescu (1997) present experimental evidence confirming this belief involving production
tasks, agents are asked to produce a sequence of fair coin tosses. They find that agents
tend to produce sequences with too many reversals and too few long runs. Bar-Hillel
and Wagenaar (1991) and Burns and Corpus (2004) provide corresponding evidence for
judgment tasks (for example, agents are asked if a given sequence of fair coin tosses) and
prediction tasks (for example, agents are asked to assign probabilities to the next coin toss
given a sequence of fair coin tosses). Rabin (2002) and Rabin and Vayanos (2010) propose
models of the LSN in discrete time and show how the LSN generates the Gambler’s fallacy.3
Our model of the LSN can be viewed as a continuous time analogue of the model in Rabin
and Vayanos (2010).

These cognitive biases are present when individual make decision in high stakes settings.
Clotfelter and Cook (1993) document the Gambler’s fallacy in lottery play and Coval and
Shumway (2005) document it in casino betting. Chen et al. (2015) provide evidence that
belief in the LSN affects the (professional) decisions of judges, loans officers, and Major
League Baseball umpires. Haigh and List (2005), Coval and Shumway (2005), and Locke
and Mann (2005) find that cognitive biases can affect trading by professional traders,

3Anecdotal evidence of the Gambler’s fallacy dated from Laplace (1951). Laplace notes that “I have
seen men, ardently desirous of having a son, who could learn only with anxiety of the births of boys in the
month when they expected to become fathers. Imagining that the ratio of these births to those of girls
ought to be the same at the end of each month, they judged that the boys already born would render more
probable the births next of girls.”

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though traders have mechanisms to limit the effects of behavioral biases in the study by Locke and Mann (2005).

The importance of cognitive biases in financial market is a topic of debate despite the evidence above. Several researchers, following Friedman (1953), argue that cognitive biases are irrelevant investors holding incorrect beliefs will lose money and not survive in the long-run. Arrow (1982), along with other authors making the case that behavioral biases should be considered as possible explanations for documented deviations from rational behavior, points to limits to arbitrage to counter Friedman’s argument. Kogan et al. (2006) show that irrational beliefs can affect prices even when irrational agents do not survive in the long-run. Borovička (2015) shows that investors with irrational beliefs can survive in the long-run when investors have recursive preferences. Freddy does not survive in the long-run in our model. However, her presence affects price for over 40 years within our simulations.

Our paper is part of the literature using behavioral models to explain important empirical irregularities. Barberis et al. (1998) and Daniel et al. (1998) model a representative agent with two cognitive biases to explain momentum and reversal (along with other irregularities) in a discrete time economy. Hong and Stein (1999) focus on the interact of two boundedly rational agents in a discrete time economy and show that it also generates momentum and reversal. We show that belief in the LSN also generates momentum, along with the equity premium and the V-shaped trading patterns, which are not discussed in those papers. Barberis et al. (2015) is closely related to our paper. They also consider a two-investors economy where one of the investors has endogenous subjective belief. The key difference between our papers is that the irrational agent forms expectation about future prices (not future dividends) by extrapolating past prices in their model. Moreover, they

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4See Barberis and Thaler (2003) and Gromb and Vayanos (2010) for more on limits to arbitrage.
use CARA utilities, an arithmetic Brownian motion dividend, and an exogenous risk-free rate.

Barberis and Xiong (2009), Ingersoll and Jin (2012), and Li and Yang (2013) show that prospect theory preferences can explain the disposition effect. Barberis and Xiong (2009) note that the choice of the reference point is crucial. Ingersoll and Jin (2012) embed prospect theory preferences in a dynamic partial equilibrium model that generates V-shaped trading patterns consistent with empirical evidence. Li and Yang (2013) show that prospect theory preferences can simultaneously generate a high equity premium, momentum, and a disposition effect. Li and Yang use an overlapping generation model with no short-selling and constant risk-free rate. We show that a parsimonious belief-based model generate qualitatively similar predictions even in the presence of short-selling and rational investors. The V-shaped trading patterns present in our model are absent from Li and Yang’s model.

Our paper is also related to the vast literature concerned with heterogeneous beliefs in financial markets. Papers here often assume that a subset of investors are either always optimistic or always pessimistic. Scheinkman and Xiong (2003), David (2008), Dumas et al. (2009), and Xiong and Yan (2010) depart from this characteristic and consider time-varying beliefs like we do. Scheinkman and Xiong (2003) and Dumas et al. (2009) focus on speculative trading and excess volatility, David (2008) addresses the equity premium, and Xiong and Yan (2010) are concerned with the failure of the expectation hypothesis. There is a fundamental difference concerning how time-variation arise between their models and mine despite the mathematical similarities. Dividend growth drifts are unobservable in their models and investors derive different information from past realizations of dividend growth. For example, Scheinkman and Xiong (2003) and Dumas et al. (2009) assume that investors over-estimate the value of a private signal that they receive. Having full
information, which is the case in our model, would lead to homogeneous beliefs in their model.

We note that Epstein (2006) develops axiomatic foundations of Non-Bayesian updating which yields beliefs akin to the Gambler’s fallacy. Epstein’s work builds on Gul and Pesendorfer’s theory of temptation, self-control, and the notion of betweenness preference. He shows that temptation can lead agents to deviate from Baye’s rule and overweight past realizations when updating their beliefs.
1. Model

1.1. Assets under Arbitrary Beliefs

We consider a continuous time infinite horizon pure exchange economy. The economy is endowed with $N$ trees, each that produces a flow of non-storable good that also serves as the numeraire. We denote the endowment processes by $D_{jt}$ and assume that they are Geometric Brownian processes on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t^j, P^j_t)$:

$$\frac{dD_{jt}}{D_{jt}} = \mu_j dt + \sigma_j dB_{jt}$$

where $D_{j0} > 0, \mu_j > 0, \sigma_j > 0$, and $B_{jt}$ is a Brownian Motion of the probability space. Let

$$D_t = \sum_{j=1}^N D_{jt}; \quad s_j = \frac{D_{jt}}{D_t}; \quad y_j = \ln D_j; \quad u_j = \ln \frac{s_j}{s_1} = y_j - y_1.$$

Moreover, let

$$\Sigma = \text{diag}(\sigma_1, \ldots, \sigma_N); \quad B = (B_1, \ldots, B_N)^t; \quad \mu = (\mu_1, \ldots, \mu_N)^t;$$

$$s = (s_1, \ldots, s_N)^t; \quad y = (y_1, \ldots, y_N)^t; \quad u = (u_1, \ldots, u_N)^t.$$

It follows that

$$\frac{dD_t}{D_t} = \mu_t dt + \sigma_t dB_t$$

where

$$\mu_t = s^t \mu; \quad \sigma_t = \sqrt{s^t \Sigma \Sigma s}; \quad \text{and} \quad dB_t = \frac{1}{\sigma} s^t \Sigma B.$$

There is a market where shares of the trees are traded. There is also a locally risk-free asset in zero-net supply that can be traded.
We assume that there is a unit mass of agents in the market. All agents have identical risk preferences (constant relative risk-aversion with coefficient $\gamma$) and time discounting (exponential at rate $\beta$). Agents choose a consumption-allocation policies to maximize their expected lifetime utility:

$$U_i = E^i \left[ \int_0^\infty e^{-\beta t} \frac{C_{i,t}^{1-\gamma}}{1-\gamma} dt \right],$$  \tag{1}

where $\beta, \gamma > 0$.

The expectation in Equation (1) is taken with respect to each agent’s subjective probability distribution of future dividend growth. The agents differ with respect to their belief about the dividend growth, even though we consider a complete information economy. We assume that agents can be put in one of two groups, 1 or 2, according to their beliefs. Members of the same group hold the same belief about each dividend growth. Beliefs of agents in Group $I$ are captured by the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t^j, P_{I,t}^j)$. A sufficient statistic for these beliefs is the Radon-Nikodyn derivative of $P_{I,t}^j$ with respect to $P_t^j$:

$$\eta_{I,t}^j = \frac{dP_{I,t}^j}{dP_t^j}; \quad \eta_{I,0}^j = 1.$$

**Assumptions**

1. $\eta_{I,t}^j$ is a strictly positive martingale with respect to $P_t^j$.

2. $$\frac{d\eta_{I,t}^j}{\eta_{I,t}^j} = \theta_{I,j}(t)dB_{jt} \quad \text{and} \quad \int_0^T \theta_{I,j}(t)dt < \infty \quad \text{almost surely, \ \forall \ T > 0}.$$

3. The variable $\theta_{I,j}$ is a stochastic process adapted to the filtration $\mathcal{F}^j$:

$$d\theta_{I,j} = \mu_{\theta_{I,j}}dt + \sigma_{\theta_{I,j}}dB_{jt}.$$
Let

\[ \theta_I = (\theta_{I,1}, \cdots, \theta_{I,N})'. \]

Define

\[ \frac{d\eta_{I,t}}{\eta_{I,t}} = \theta_I'dB_t \quad \eta_{I,0} = 1. \]

Then, \( \eta_{I,t} \) is the Radon-Nikodym derivative of Agent I’s subjective probability distribution with respect to the true probability distribution. We have

\[ \eta_{I,t} = \prod_{j=1}^{N} \eta_{I,t}^j. \]

**Proposition 1.** Survival and long-term price impact are equivalent within our model. Moreover, Agent I does not survive if and only if

\[ \lim_{t \to \infty} \int_0^t \left[ \|\theta_I(s)\|^2 - \|\theta_{-I}(s)\|^2 \right] ds = \infty \]

where

\[ \| (x_1, \cdots, x_n) \| = \sqrt{x_1^2 + \cdots + x_n^2}. \]

We can interpret \( \theta_I \) as the error made in estimating the expected dividend growth. The proposition above says that an agent does not survive if her mean-squared error grows too fast relative to the other agent.
1.2. Beliefs in the Law of Small Numbers in Continuous Time

The dynamics of $\theta_{I,j}$ determine the belief of Agent $I$. We motivate our choice for these dynamics for a subset of agents in our model using the belief in the Law of Small Numbers.\(^5\)

Belief in the LSN is the mistaken belief that a small sample drawn from a population should be representative of that population. Experimental evidence shows that individuals hold this belief (see \cite{Tversky and Kahneman 1971}, \cite{Rapoport and Budescu 1997}, \cite{Bar-Hillel and Wagenaar 1991}, \cite{Burns and Corpus 2004}, and references therein). Moreover, belief in the LSN affects individuals, decision-making process in high stakes settings (see \cite{Clotfelter and Cook 1993}, \cite{Coval and Shumway 2005}, \cite{Haigh and List 2005}, \cite{Coval and Shumway 2005}, \cite{Locke and Mann 2005}, \cite{Chen et al. 2015}, and references therein).

The following example illustrates how belief in the LSN influence the way agents update their beliefs. Suppose that Freddy knows that a coin is fair. Then belief in the LSN leads Freddy to think that any sequence of independent coin tosses using this fair coin is representative of the population of fair coins. That is, Freddy believes that any sequence of coin tosses will have a ratio of number of Heads to number of Tails that is closed to 50%. Assume that Freddy observes four independent coin tosses and is asked about her beliefs concerning the fifth coin toss. If the first four tosses resulted in four Heads, then belief in the LSN leads Freddy to assign a probability higher than 50% to the event \{The fifth coin toss is a Head\}. This belief about the fifth coin toss rationalizes the belief that the random sample of the five coin tosses will be representative of the fair coins’ population. If the first four tosses resulted in four Heads, then Freddy also to assign a probability higher than 50% to the same event. However, the probability assigned by Freddy in the three Heads case is not greater than than in the four Heads case. This monotonicity results from the fact that the three Heads sample is more representative of

\(^5\)See \cite{Scheinkman and Xiong 2003} for dynamics motivated by overconfidence.
the population relative to the four Heads sample.

We now present a continuous-time model of the belief in the LSN within a Lucas economy with a single tree, which can be easily extended to the Lucas Orchard we consider. Freddy knows that the expected instantaneous dividend growth of the tree is $\mu dt$. Thus, Freddy’s belief about the tree’s dividend growth at time $t$ is a function of the history up to time $t$ of deviations from $\mu$. We use the following state variable as a proxy for the history:

$$\theta_t \equiv -b \int_0^t e^{-\kappa(t-s)} \frac{1}{\sigma} \left[ \frac{dD_s}{D_s} - \mu dt \right] = -b \int_0^t e^{-\kappa(t-s)} dB_s, \quad \theta_0 = 0, \quad (2)$$

where $\kappa, b \geq 0$. The integral in Equation (2) is the aggregate of the realized deviations from the expected growth rate $\mu$. The term $\kappa$ captures the fact that recent deviations are more salient. The term $b$ measures the degree of the belief in the LSN. The dynamics of $\theta$ are

$$d\theta_t = -\kappa \theta_t dt + b dB_t.$$

It follows that Freddy believes that the dividend growth process is

$$\frac{dD_t}{D_t} = (\mu + \sigma \theta_t) dt + \sigma d\hat{B}_t,$$

where $\hat{B}_t$ is the Brownian motion with respect to Freddy’s subjective probability and

$$dB_t = d\hat{B}_t + \theta_t.$$

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That is, Freddy believes that the expected dividend growth at time $t$ is

$$\mu + \sigma \theta_t.$$  

The state variable $\theta_t$ satisfies

$$\theta_u - \theta_t = -b(B_u - B_t).$$

when $\kappa = 0$. Thus, Freddy’s bias when estimating the expected dividend growth is a scalar (b) times the negative of the sum (or integral) of past dividend growth shocks ($dB$), consistent with belief in reversal. In general,

$$\theta_u - \theta_t = -b(B_u - B_t) - b\kappa \int_t^u \theta_s ds, \quad u > t$$

Therefore, Freddy believes in reversal, but her beliefs are persistent: The belief at time $u > t$ depends on the sequence of shocks between $t$ and $u$ but also of the history of shocks up to time $t$. A positive shock at time $t$, following a long history of negative shocks does not necessarily result in Freddy believing the expected dividend growth at time $u > t$ is lower than that at time $t$.

2. Asset Pricing with Belief in the Law of Small Numbers

2.1. Equilibrium

The market is dynamically complete. Therefore it is enough to consider the social planner’s problem. The social planner’s problem is

$$\sup_{C_{1,t} + C_{2,t} = D_t} E^1 \left\{ \frac{e^{-\beta t}}{1 - \gamma} \left[ \lambda C_{1,t}^{1-\gamma} + (1 - \lambda)\xi_t C_{2,t}^{1-\gamma} \right] \right\}$$

$$\tag{3}$$
where with take the expectation with respect to Agent 1’s subjective probability distribution and 
\[ \xi_t \equiv \frac{\eta_{2,t}}{\eta_{1,t}} \equiv \prod_{j=1}^{N} \xi_{j,t}; \quad \text{where} \quad \xi_{j,t} \equiv \frac{\eta_{j,t}^2}{\eta_{j,t}^i}. \]
The first order conditions yield
\[ \lambda C_{1,t}^{-\gamma} = (1 - \lambda)\xi_t C_{2,t}^{-\gamma}. \]
Let
\[ \nu_{I,t} = \frac{C_{I,t}}{D_t} \]
be Agent I’s consumption share. The investors’ individual optimization problems imply that the following is a state price density:
\[ \pi_t = \lambda e^{-\beta t} \nu_{1,t}^{-\gamma} D_t^{-\gamma} = (1 - \lambda)\xi_t e^{-\beta t} \nu_{2,t}^{-\gamma} D_t^{-\gamma}. \]
Let
\[ \hat{\pi}_{1,t} = \lambda e^{-\beta t} D_t^{-\gamma}; \quad \hat{\pi}_{2,t} = (1 - \lambda)e^{-\beta t} \xi_t D_t^{-\gamma}; \quad \text{and} \quad A_t = \left( \frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}} \right)^{-1/\gamma}. \]
The optimal consumption shares are
\[ \nu_{1,t} = \frac{A_t}{1 + A_t} = \frac{1}{1 + (\alpha \xi_t)^{1/\gamma}} \quad \text{and} \quad \nu_{2,t} = 1 - \nu_{1,t}, \]
where \( \alpha = (1 - \lambda)/\lambda. \)

The state price density is
\[ \pi_t = \lambda e^{-\beta t} D_t^{-\gamma} \left[ 1 + (\alpha \xi_t)^{1/\gamma} \right]^{\gamma}. \]
Applications of Ito’s Lemma yield the following Proposition:

**Proposition 2.** The instantaneous risk-free rate is

\[ r_{f,t} = r_{f,t}^b + \sum_{j=1}^{N} r_{f,t}^{h_j} \]  

(4)

where

\[ r_{f,t}^b = \gamma \mu_t + \beta - \frac{\gamma(1 + \gamma)}{2} \sum_{j=1}^{N} s_{jt}^2 \sigma_{jt}^2. \]  

Benchmark: Only rational beliefs

\[ r_{f,t}^{h_j} = \nu_{2,t}(\theta_{2,j}(t) - \theta_{1,j}(t)) \left[ \gamma s_{jt} \sigma_j + \frac{\gamma - 1}{2\gamma} v_{1,t} (\theta_{2,j}(t) - \theta_{1,j}(t)) + \theta_{1,j}(t) \right]. \]

The market price of risk of the \( j \)th tree (as perceived by Agent 1) is

\[ \sigma_{\pi}^j = \gamma \sum_{j=1}^{N} \left[ s_{jt} \sigma_j - \frac{\nu_{2,t}}{\gamma} (\theta_{2,j}(t) - \theta_{1,j}(t)) \right]. \]  

(5)

The market price of risk the \( j \)th tree as perceived by Agent 1 has two components. The first component is the standard compensation for bearing aggregate risk. The second component is decreasing (and linear) in both the consumption share of Agent 2 and the difference in beliefs \( \theta_{2,j} - \theta_{1,j} \). To understand this relation, note that Agent 1’s consumption share satisfies

\[ \frac{d \nu_{1,t}}{\nu_{1,t}} dB_{jt} = -\frac{\nu_{2,t}}{\gamma} (\theta_{2,j}(t) - \theta_{1,j}(t)) dt. \]

A positive shock to the \( j \)th tree’s dividends results in lower consumption share for Agent
1 when \((\theta_{2,j} - \theta_{1,j})\) is positive. Intuitively, Agent 2 is more optimistic about the \(j\)th tree’s future dividends relatively to Agent 1 when \((\theta_{2,j} - \theta_{1,j}) > 0\). Thus, Agent 1 becomes relatively wealthier following positive shock to the \(j\)th tree’s dividends because these are consistent with the optimism, which implies lower consumption share for Agent 2. Therefore, when \((\theta_{2,j} - \theta_{1,j})\) is positive, a positive shock to the \(j\)th tree’s dividends is correlated with lower consumption share for Agent 2 and as a result risk associated with the \(j\) tree carries negative price.

The instantaneous risk-free rate is the sum of the instantaneous risk-free rate in the corresponding Lucas economy without Freddy and a linear-quadratic term in \(\theta_t\). The state variable \(\theta_t\) has two effects on the risk-free rate. First, Freddy expects higher (lower) dividends in the future and thus has lower (higher) future marginal utility when \(\theta_t\) is positive (negative). Thus, the risk-free rate is higher (lower) when \(\theta_t\) is positive (negative) because Freddy consumes more (less) and saves less (more) today. Second, the two agents use the credit market for risk-sharing purposes when \(\gamma > 1\), and the need for the credit market increase with \(|\theta_t|\). Thus, higher values of \(|\theta_t|\) yield higher interest rates. The two effects combine when \(\theta_t\) is positive. The first effect dominates for small negative values of \(\theta_t\) while the second effect dominates for large negative values of \(\theta_t\). Figure 1 illustrates the relation between the instantaneous risk-free rate and \(\theta_t\) for different values of \(\xi_t\) (recall that Freddy’s consumption share is an increasing function of \(\xi\)).

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\(^7\text{See }\text{Longstaff and Wang (2012)}\) for a discussion of the credit market’s role for risk-sharing purposes.
Figure 1: **Instantaneous Risk-free Rate.** We plot the instantaneous risk-free rate as a function of $\theta_t$ for different values of $\xi_t$. The believer in the Law of Small Number (Freddy) expects dividend growth to be the true dividend growth plus $\sigma \theta_t$. $\xi_t$ is the Radon-Nikodym derivative of Freddy’s subjective probability density function with respect to the true density function. The baseline case is the interest rate in the corresponding economy where Freddy is absent.

2.2. Pricing Securities and Portfolio Allocation

Let $P_\alpha$ be the price of an asset paying dividends

$$D_j^\alpha = \prod D_j^{\alpha_j}, \text{ where } \alpha = (\alpha_1, \cdots, \alpha_N).$$

Let

$$D^\alpha = D^\alpha, \text{ where } \alpha = \sum \alpha_j.$$

Then,

$$\frac{P_\alpha}{D_j^\alpha} = E_t \left\{ \int_0^\infty e^{-\beta(u-t)} \left( \frac{1 + \alpha^{1/\gamma} \xi_t^{1/\gamma}}{1 + \alpha^{1/\gamma} \xi_t^{1/\gamma}} \right)^\gamma \left( \frac{D_u}{D_t} \right)^{-\gamma} \frac{D_j^\alpha}{D_j^{\alpha_j}} du \right\}.$$
Computing this ratio will require computing expectations of the form

\[ H(t) = \mathbb{E}\left\{ \left( \frac{1 + \alpha \xi_1^{1/\gamma}}{1 + \alpha \xi_t^{1/\gamma}} \right)^m \left( \frac{D_u}{D_t} \right)^n \prod \left( \frac{D_{j,u}}{D_{j,t}} \right)^{\alpha_j} \right\} \] (6)

Pricing the assets that we are interested in will require computing integrals of the form

\[ f(t, D_t^{\alpha}, \xi_t, \theta; n, m) = \int_0^\infty e^{-\beta \tau} \mathbb{E}_t \left\{ \left( \frac{1 + \alpha \xi_1^{1/\gamma}}{1 + \alpha \xi_t^{1/\gamma}} \right)^m \left( \frac{D_u}{D_t} \right)^n \prod \left( \frac{D_{j,u}}{D_{j,t}} \right)^{\alpha_j} \right\} d\tau \] (7)

for \( n, \alpha_1, \cdots, \alpha_N \in \mathbb{R} \), which in turn requires computing expectations of the form

\[ H(\tau, D_t, \xi_t, \theta; n, m) = \mathbb{E}_t \left\{ \left( \frac{\xi_u}{\xi_t} \right)^m \left( \frac{D_u}{D_t} \right)^n \prod \left( \frac{D_{j,u}}{D_{j,t}} \right)^{\alpha_j} \right\}. \] (8)

We follow Dumas et al. (2009) and show in the Appendix that \( H \) is

\[ H(t, u, D_t^{\alpha}, \xi_t, \theta; n, m) = \exp \left\{ n \left( \mu - \frac{1}{2} \sigma^2 \right) \tau + \frac{n^2}{2} \sigma^2 \tau \right\} \exp \left\{ K_N(\theta, \tau) \right\} \] (9)

where

\[ K_N(\theta, \tau) = \sum_{j=1}^N \left[ A_{j,0}(\tau) + \theta_j A_{j,1}(\tau) + \theta_j^2 A_{j,2}(\tau) \right] \]

and the functions \( A_{j,0}, A_{j,1}, \) and \( A_{j,2} \) are defined in the Appendix. Assume that \( n \in \mathbb{N} \).

Then

\[ f(t, D_t^{\alpha}, \xi_t, \theta; n, m) = \int_0^\infty e^{-\beta \tau} \left( 1 + \alpha \xi_1^{1/\gamma} \right)^{-m} \sum_{j=0}^n \binom{n}{j} \left( \xi_t \right)^{j/\gamma} H(t, u, D_t^{\alpha}, \xi_t, \theta; n, j/\gamma) d\tau. \] (10)

The dynamics of \( f \) can be obtained by applying Itô’s lemma. We use the following
The following proposition gives the price to dividend ratio of a claim on the Lucas tree, each agent’s equilibrium wealth and optimal portfolio:

**Proposition 3.** The price to dividend ratio is

\[
\frac{P_\alpha}{D_\alpha^j} = f_\gamma(t, D_t^\alpha, \xi_t, \theta_t).
\]

The rational agent wealth to consumption ratio is

\[
\frac{W_{1,t}}{C_{1,t}} = f_{\gamma-1}(t, D_t^1, \xi_t, \theta_t).
\]

Freddy’s wealth is

\[
W_{2,t} = P_t - W_{1,t}.
\]

Let \((M_{i,t}, N_{i,j,t})\) the Player’s portfolio, where \(M_{i,t}\) is the amount of money invested in the risk-free asset and \(N_{i,j,t}\) is the number of shares of the \(j\)th risky asset she holds. The equilibrium number of shares of the risky asset held by the agents at time \(t\) are

\[
N_{2,t} = 1 - N_{1,t}; \quad \text{and} \quad N_{1,t} = \frac{\sigma W_{1,t}}{\sigma P_t'} = \frac{W_{1,t} \sigma - \nu_2 t \frac{\theta_t}{\gamma} + \frac{\sigma f_{\gamma-1}}{f_{\gamma}}}{\sigma + \frac{\sigma f_{\gamma}}{f_{\gamma}}}. 
\]

\[
M_{2,t} = -M_{1,t}; \quad \text{and} \quad M_{1,t} = W_{1,t} - N_{1,t} P_t.
\]
The number of shares of the risky asset held by Freddy takes the form

\[ N_{2,t} = A_1 \theta_t + A_0 \]

where \( A_0, A_1 \geq 0 \). Thus, the direct effect of \( \theta_t \) on \( N_{2,t} \) is positive. The implication is that Freddy increases (decreases) her share of the risky asset between time \( t \) and time \( u > t \) if \( \theta_u - \theta_t \) is positive (negative), *Ceteris paribus*. This relation does not directly translate to one where \( \theta_t \) is replaced with dividend growth. It follows from \( \theta_t \)'s Ito’s representation that

\[ \theta_u - \theta_t = -b(B_u - B_t) - \kappa b \int_t^u \theta_s ds. \]

Thus, having

\[ B_u - B_t > 0 \]

does not necessarily imply that

\[ \theta_u - \theta_t < 0 \]

when \( \kappa \neq 0 \). However, the likelihood that \( \theta_u - \theta_t \) have opposite signs increases with the magnitude of \( |B_u - B_t| \). Hence, we conclude that it is more likely that Freddy increases (decreases) her share of the risky asset between time \( t \) and time \( u > t \) if \( B_u - B_t \) is negative (positive) and \( |B_u - B_t| \) is large, *Ceteris paribus*. In terms, of returns, this conclusion translates into a higher likely that Freddy increases (decreases) her share of the risky asset between time \( t \) and time \( u > t \) if the risky asset’s return is negative (positive) and large in magnitude, *Ceteris paribus*. This conclusion holds under the assumption that both \( A_0 \) and \( A_1 \) remain constant when we vary \( \theta_t \), which is not true in equilibrium. We confirm that the conclusion holds in equilibrium in Section 2.4 using simulations.
2.3. Returns and Excess Volatility

We examine our model’s implications for the moments of the risky asset’s return. Let \( R_t \) be the stock’s cumulative return. Then

\[
dR_t \equiv \frac{dP_t + D_t dt}{P_t} = \left( \mu + \frac{1}{f_{\gamma}} + \frac{\mu f_{\gamma} + \sigma \sigma f_{\gamma}}{f_{\gamma}} \right) dt + \left( \sigma + \frac{\sigma f_{\gamma}}{f_{\gamma}} \right) dB_t. \tag{11}
\]

The expected return has the form the standard form

\[
\text{dividend drift + dividend yield}
\]

plus a term accounting for the variability of the price to dividend ratio. Similarly, the return volatility is equal to the fundamental volatility plus a term accounting for the volatility of the price to dividend ratio.

Figure 2 shows that both the instantaneous return and the equity premium are higher in our model relative to their values in the corresponding economy where Freddy is absent.\footnote{Abel et al. (1989) first noted that heterogeneous beliefs can lead to a higher equity premium.} Freddy’s beliefs add to the uncertainty in the economy because they are time-varying and affect Freddy’s subjective probability about the economy. This additional uncertainty leads to lower prices and thus higher dividend yield and higher returns.

Figure 3 shows that both the instantaneous return and the equity premium are higher in our model relative to their values in the corresponding economy where Freddy is absent.\footnote{Abel et al. (1989) first noted that heterogeneous beliefs can lead to a higher equity premium.} Freddy’s beliefs add to the uncertainty in the economy because they are time-varying and affect Freddy’s subjective probability about the economy. This additional uncertainty leads to lower prices and thus higher dividend yield and higher returns.
Figure 2: **Instantaneous Market Return and Equity Premium.** We plot the instantaneous market return (left) and the equity premium (right) as functions of $\theta_t$ for different values of $\xi_t$. The believer in the Law of Small Number (Freddy) expects dividend growth to be the true dividend growth plus $\sigma \theta_t$. $\xi_t$ is the Radon-Nikodym derivative of Freddy’s subjective probability density function with respect to the true density function. The baseline cases are the respective quantities in the corresponding economy where Freddy is absent.

The market portfolio’s volatility is

$$\sigma_M = \sigma + \frac{\sigma \gamma}{\gamma \gamma}.$$

We refer to the term

$$\frac{\sigma \gamma}{\gamma \gamma}$$

as the excess volatility.

Figure 3 shows that the excess volatility

1. Positive for values of $\theta_t$ within one standard deviation of its mean, the mean which is zero.
2. Concave in shocks to dividend growth.
Figure 3: Excess Volatility. We plot the excess volatility as functions of $\theta_t$ for different values of $\xi_t$. The believer in the Law of Small Number (Freddy) expects dividend growth to be the true dividend growth plus $\sigma \theta_t$. $\xi_t$ is the Radon-Nikodym derivative of Freddy’s subjective probability density function with respect to the true density function. The baseline cases are the respective quantities in the corresponding economy where Freddy is absent.

3. Counter-cyclical when positive.
2.4. Simulations

We simulate 100 economies each over a 50 years period and compute relevant quantities for each economy.

**Definition 1.** *Returns exhibit momentum if*

\[
E[R_u|R_t - R_s > a] - E[R_u|R_t - R_s < a] > 0,
\]

*where* *a* > 0.*

**Definition 2.** Freddy exhibits the disposition effect if

\[
N_{2,t}|R_t - R_s > a < N_{2,s} < N_{2,t}|R_t - R_s < a
\]

*where* *a* > 0. Equivalently, Freddy exhibits the disposition effect if

\[
N_{1,t}|R_t - R_s > a > N_{1,s} > N_{1,t}|R_t - R_s < a
\]

Both *u − t* and *a* in the definition above should be “small enough”. What is small enough? I don’t know yet.

2.4.1. State Variables

Figure 4 (b) shows that Freddy’s influence vanishes in the long-run.
2.4.2. Consumption

Figure 5 (a) shows that Freddy does not survive in the long run.
2.4.3. Risk-Free rate and Market Price of Risk

Figure 6 shows that the risk-free rate is higher in our model relative to the baseline without Freddy. Both the risk-free rate and the market price of risk vary over time.

An important remark is that both the risk-free rate and the market price of risk in our model are different from their respective values in the baseline case even after both Freddy’s consumption and $\xi_t$ become negligible.
2.4.4. Disposition Effect

We examine Freddy’s trading patterns. First we examine the likelihood of Freddy selling (buying) the risky asset conditional on changes in returns. Figure 7 shows that Freddy is more likely to sell the risky asset following an increase in returns relative to a decrease in returns, with both probabilities been positive. The opposite holds for buys. These patterns are consistent with the empirical findings of Ben-David and Hirshleifer (2012).

Next we consider Freddy’s portfolio holding following increases/decreases in returns. Every months, we rank stocks into quintiles based on the difference between the present month and the previous month returns and compute the average percentage change in holding of the risky asset relative to the previous month within each group. The average difference between the average of the highest returns group and the lowest returns group is

\[-0.0179,\]
which is negative. Thus, on average,

\[ N_{2,t|R_t - R_s > b} < N_{2,t-1} < N_{2,t|R_t - R_s < a} \]

for two positive numbers \( a \) and \( b \). That is, Freddy reduces (increases) her holding of the risky asset following an increase (decrease) in returns.

Both the probability of selling additional shares and the percentage changes in holding
of the risky asset following increases/decreases in returns suggest that Freddy exhibits the disposition effect. We test directly for the disposition effect. We follow Odean (1998) and compute both the percentage of realized gains (PRG) and the percentage of realized losses (PRL) every months. Table 1 shows that

\[ \text{PRG} > \text{PRL} \]

every month. Therefore, Freddy exhibits the disposition effect.

Table 1: Disposition Effect. The market is simulated 100 times, each over a 50 years period, and the portfolio of each agent is computed. We define the percentage of realized gains (PRG) as the number of stocks Freddy liquidates whose returns have increased divided by the total number of stocks whose returns have increased. We use the corresponding definition for the percentage of realized losses (PRL). Each month, we compute the difference between PRG and PRL and report the summary statistics of this series.

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Std Dev</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>PRG - PRL</td>
<td>0.41</td>
<td>0.1</td>
<td>0.055</td>
<td>0.70</td>
</tr>
</tbody>
</table>
2.4.5. Momentum

Trading by Freddy causes prices to deviate from the rational benchmark. We study the asset pricing implications of the LSN. An obvious starting point is momentum, given the V-shaped patterns associated with Freddy’s trading.

Table 2 shows that our model replicates the return momentum documented empirically. The momentum is stronger in the first half of our sample, when Freddy’s consumption is a large percentage of the dividend flow, relative to the second half. Therefore, the momentum is driven by the belief in the LSN.

Although we do not attempt to match empirical moment, we note that the momentum in our simulated data is low relative to the empirical momentum. Thus, our model does not explain the majority of the empirical momentum. This result is consistent with the findings of Birru (2015). Birru shows that momentum is present in the market even when the disposition effect is absent among retail investors.

Table 2: Momentum. The market is simulated 100 times, each over a 50 years period and monthly returns are computed. Each month, we group stocks into deciles based on the cumulative returns over the previous twelve months. The table shows descriptive statistics of the differences of average returns between the high decile and the low decile for the month following the formation of portfolios.

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Std Dev</th>
<th>T-Stat</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>Full Sample</td>
<td>0.0018</td>
<td>0.007</td>
<td>6.24</td>
<td>−0.019</td>
<td>0.025</td>
</tr>
<tr>
<td>First Half</td>
<td>0.003</td>
<td>0.007</td>
<td>6.67</td>
<td>−0.017</td>
<td>0.025</td>
</tr>
<tr>
<td>Second Half</td>
<td>0.0007</td>
<td>0.006</td>
<td>1.9</td>
<td>−0.019</td>
<td>0.018</td>
</tr>
</tbody>
</table>
Appendix A. Conditional Expectations

Each month $t$, I will classify each economy as H (L) if

$$R_t - R_{t-h} > 0 \quad (R_t - R_{t-h} < 0)$$

where $h > 0$ is an integer multiple of one month.

For a given stochastic process $X_t$, I define the conditional expectation

$$E [X_{t+h'} - X_t | R_t - R_{t-h} > 0]$$

as the average of $X_{t+h'} - X_t$ among the economies that are in the H group at time $t$. Other conditional expectations are defined along the same lines.
Appendix B. Ito’s Results

\[
\frac{dD^{-\gamma}}{D^{-\gamma}} = \left[-\gamma \mu + \frac{1}{2} \gamma (\gamma + 1) \sigma^2\right] dt - \gamma \sigma dB
\]

\[
\frac{d(\alpha \xi)^{1/\gamma}}{(\alpha \xi)^{1/\gamma}} = \frac{1}{2\gamma^2} (\theta_2 - \theta_1) \left[(\theta_2 - \theta_1) - \gamma(\theta_2 + \theta_1)\right] dt + \frac{1}{\gamma} (\theta_2 - \theta_1) dB
\]

\[
\frac{dv_1}{\nu_1} = \frac{d \left[\frac{1 + (\alpha \xi)^{1/\gamma}}{1 + (\alpha \xi)^{1/\gamma}}\right]^{-1}}{\nu_1} = -\nu_1 d(\alpha \xi)^{1/\gamma} + \nu_1^2 \left(d(\alpha \xi)^{1/\gamma}\right)^2
\]

\[
= -\nu_2 \left\{ \frac{1}{2\gamma^2} (\theta_2 - \theta_1) \left[(\theta_2 - \theta_1) - \gamma(\theta_2 + \theta_1)\right] dt + \frac{1}{\gamma} (\theta_2 - \theta_1) dB \right\} + \nu_2^2 \frac{1}{\gamma^2} (\theta_2 - \theta_1)^2 dt
\]

\[
= \left\{ -\nu_2 \left[ \frac{1}{2\gamma^2} (\theta_2 - \theta_1) \left[(\theta_2 - \theta_1) - \gamma(\theta_2 + \theta_1)\right] + \nu_2^2 \frac{1}{\gamma^2} (\theta_2 - \theta_1)^2 \right] dt - \nu_2 \frac{1}{\gamma} (\theta_2 - \theta_1) dB \right\}
\]

\[
= \frac{\nu_2}{2\gamma^2} (\theta_2 - \theta_1) \left[\gamma(\theta_2 + \theta_1) + (\nu_2 - \nu_1)(\theta_2 - \theta_1)\right] dt - \nu_2 \frac{1}{\gamma} (\theta_2 - \theta_1) dB
\]

\[
\frac{dv_{1^{-\gamma}}}{\nu_{1^{-\gamma}}} = \left\{ -\frac{\nu_2}{2\gamma} (\theta_2 - \theta_1) \left[\gamma(\theta_2 + \theta_1) + (\nu_2 - \nu_1)(\theta_2 - \theta_1)\right] + \frac{\gamma + 1}{2\gamma} \nu_2^2 (\theta_2 - \theta_1)^2 \right\} dt + \nu_2 (\theta_2 - \theta_1) dB
\]

\[
= \frac{\nu_2}{2\gamma} (\theta_2 - \theta_1) \left\{ -\left[\gamma(\theta_2 + \theta_1) + (\nu_2 - \nu_1)(\theta_2 - \theta_1)\right] + (\gamma + 1) \nu_2 (\theta_2 - \theta_1) \right\} dt + \nu_2 (\theta_2 - \theta_1) dB
\]

\[
= \frac{\nu_2}{2\gamma} (\theta_2 - \theta_1) \left[\gamma(\theta_2 + \theta_1) + (\nu_2 + \nu_1)(\theta_2 - \theta_1)\right] dt + \nu_2 (\theta_2 - \theta_1) dB
\]

\[
= \frac{\nu_2}{2\gamma} (\theta_2 - \theta_1) \left[\frac{1 - \gamma}{\gamma} (\theta_2 - \theta_1) - 2\gamma \theta_1\right] dt + \nu_2 (\theta_2 - \theta_1) dB
\]

\[
\frac{d\pi}{\pi} = -\left\{ \beta + \gamma \mu - \frac{1}{2} \gamma (\gamma + 1) \sigma^2 - \frac{\nu_2}{2\gamma} (\theta_2 - \theta_1) \left[(1 - \gamma)(\theta_2 - \theta_1) - 2\gamma \theta_1\right] + \gamma \sigma \nu_2 (\theta_2 - \theta_1) \right\} dt
\]

\[
= -\left\{ \beta + \gamma \mu - \frac{1}{2} \gamma (\gamma + 1) \sigma^2 + \nu_2 (\theta_2 - \theta_1) \left[\gamma \sigma + \frac{\gamma - 1}{2\gamma} (\theta_2 - \theta_1) + \theta_1\right] \right\} dt
\]

\[
= -\left[\gamma \sigma - \nu_2 (\theta_2 - \theta_1) \right] dB
\]

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Appendix C. Pricing Assets

Let \( P_\alpha \) be the price of an asset paying dividends

\[
D_j^\alpha \equiv \prod D_j^{\alpha_j}, \quad \text{where} \quad \alpha = (\alpha_1, \ldots, \alpha_N).
\]

Let

\[
D^\alpha \equiv D^\alpha, \quad \text{where} \quad \alpha = \sum \alpha_j.
\]

Then,

\[
\frac{P_\alpha}{D_j^\alpha} = \mathbb{E}_t \left\{ \int_0^\infty e^{-\beta(u-t)} \left( \frac{1 + \alpha^{1/\gamma} \xi^{1/\gamma}_u}{1 + \alpha^{1/\gamma} \xi^{1/\gamma}_t} \right)^\gamma \left( \frac{D_u}{D_t} \right)^{-\gamma} \frac{D_{j,u}}{D_{j,t}} du \right\}.
\]

Computing this ratio will require computing expectations of the form

\[
H() = \mathbb{E} \left\{ \left( \frac{1 + \alpha^{1/\gamma} \xi^{1/\gamma}_u}{1 + \alpha^{1/\gamma} \xi^{1/\gamma}_t} \right)^m \left( \frac{D_u}{D_t} \right)^n \prod \left( \frac{D_{j,u}}{D_{j,t}} \right)^{\alpha_j} \right\} \quad (C.1)
\]

Appendix C.1. No Bias, One Tree

This case is the standard Lucas tree:

\[
H() = \mathbb{E} \left\{ \left( \frac{D_u}{D_t} \right)^n \right\} = \exp \left\{ n \left( \mu - \frac{1}{2} \sigma^2 \right) \tau + \frac{n^2}{2} \sigma^2 \tau \right\}.
\]
Appendix C.2. No Bias, Two Trees

This case a special case of Martin (2013):

\[ H() = E \left\{ \frac{D_u}{D_t} - n \left( \frac{D_{1,u}}{D_{1,t}} \right)^\alpha \left( \frac{D_{2,u}}{D_{2,t}} \right)^\beta \right\}. \]

We will solve for the expectation

\[ H() = E \left\{ \frac{e^{\alpha \tilde{y}_{1,u} + \beta \tilde{y}_{2,u}}}{(e^{\tilde{y}_{1,u} + y_{1,t}} + e^{\tilde{y}_{2,u} + y_{2,t}})^n} \right\} \]

\[ = E \left\{ e^{-n \frac{\tilde{y}_{1,u} + y_{1,t} + \tilde{y}_{2,u} + y_{2,t}}{2}} \frac{e^{\alpha \tilde{y}_{1,u} + \beta \tilde{y}_{2,u}}}{(e^{\tilde{y}_{1,u} + y_{1,t} - (\tilde{y}_{2,u} + y_{2,t})/2} + e^{-\tilde{y}_{1,u} + y_{1,t} - (\tilde{y}_{2,u} + y_{2,t})/2})^n} \right\} \]

\[ = E \left\{ e^{-n \frac{y_{1,t} + y_{2,t}}{2}} \frac{e^{(\alpha-n/2)\tilde{y}_{1,u} + (\beta-n/2)\tilde{y}_{2,u}}}{(2 \cosh \left[ \frac{\tilde{y}_{1,u} + y_{1,t} - (\tilde{y}_{2,u} + y_{2,t})}{2} \right])^n} \right\} \]

\[ = E \left\{ e^{-n \frac{y_{1,t} + y_{2,t}}{2}} \frac{e^{(\alpha-n/2)\tilde{y}_{1,u} + (\beta-n/2)\tilde{y}_{2,u}}}{(2 \cosh \left[ -\frac{\tilde{y}_{1,u} + y_{1,t} - (\tilde{y}_{2,u} + y_{2,t})}{2} \right])^n} \right\} \]

\[ = e^{-n \frac{y_{1,t} + y_{2,t}}{2}} E \left\{ e^{(\alpha-n/2)\tilde{y}_{1,u} + (\beta-n/2)\tilde{y}_{2,u}} \int_{-\infty}^{\infty} e^{i[y_{2,u} + y_{2,t} - (\tilde{y}_{1,u} + y_{1,t})]z} F(z) dz \right\} \]

\[ = e^{-n \frac{y_{1,t} + y_{2,t}}{2}} \int_{-\infty}^{\infty} e^{i[y_{2,u} - y_{1,t}]z} E \left\{ e^{(\alpha-n/2-iz)\tilde{y}_{1,u} + (\beta-n/2+iz)\tilde{y}_{2,u}} \right\} F(z) dz \]

\[ = e^{-n \frac{y_{1,t} + y_{2,t}}{2}} \int_{-\infty}^{\infty} e^{i[y_{2,u} - y_{1,t}]z} e^{(\alpha-n/2-iz; \beta-n/2+iz)} F(z) dz \]
Appendix C.3. No Bias, N Trees

This case is the model considered by Martin (2013):

\[ H() = E \left\{ \left( \frac{D_u}{D_t} \right)^{-n} \prod \left( \frac{D_{j,u}}{D_{j,t}} \right)^{\alpha_j} \right\}. \]

We are interested in the expectation:

\[ H() = \mathbb{E} \left\{ e^{\sum \alpha_j \tilde{y}_{j,u}} \left( \sum e^{\tilde{y}_{j,u} + y_{j,t}} \right)^{-n} \right\} \]

\[ = \mathbb{E} \left\{ e^{\alpha' \tilde{y}_u} \left( \sum e^{\tilde{y}_{j,u} + y_{j,t}} - n' \right)^{-n} \right\} \]

\[ = \mathbb{E} \left\{ e^{\alpha' \tilde{y}_u - n' (\tilde{y}_u + y_t)/N} \left( \sum e^{\tilde{y}_{j,u} + y_{j,t}} - 1' (\tilde{y}_u + y_t)/N \right)^{-n} \right\} \]

\[ = \mathbb{E} \left\{ e^{\alpha' \tilde{y}_u - n' (\tilde{y}_u + y_t)/N} \int e^{i z' x} F(z) d z \right\} \]

\[ = e^{-n' y_t/N} \int \mathbb{E} \left\{ e^{(\alpha - n/N) \tilde{y}_u} e^{i z' x} \right\} F(z) d z \]

\[ = e^{-n' y_t/N} \int \mathbb{E} \left\{ e^{(\alpha - n/N) \tilde{y}_u} e^{i z' (Q(\tilde{y}_u + y_t))} \right\} F(z) d z \]

\[ = e^{-n' y_t/N} \int e^{i z' (Q y_t)} \mathbb{E} \left\{ e^{(\alpha - n/N + i Q' z') \tilde{y}_u} \right\} F(z) d z \]

\[ = e^{-n' y_t/N} \int e^{i z' (Q y_t)} e^{c(\alpha - n/N + i Q' z)} F(z) d z \]

where

\[ Q \equiv (N I - 1 \cdot 1')_{N \times (N-1)} \quad \text{and} \quad x \equiv Q(\tilde{y}_u + y_t). \]
Appendix C.4. One Freddy, One Tree

This case is similar to the model considered by Dumas et al. (2009):

\[ H() = \mathbb{E}\{D_u^{-n}\xi^m_u\}. \]

I obtained this expectation following the approach of Dumas et al. (2009). By the Feynman-Kac formula, \( H \) satisfies the PDE

\[
0 = \frac{\partial H}{\partial t} + \mu D \frac{\partial H}{\partial D} - \kappa \theta \frac{\partial H}{\partial \theta} + \frac{1}{2} \left( \sigma D \right)^2 \frac{\partial^2 H}{\partial D^2} + \frac{1}{2} (\xi \theta)^2 \frac{\partial^2 H}{\partial \xi^2} + \frac{1}{2} b^2 \frac{\partial^2 H}{\partial \theta^2} + \sigma \theta D \frac{\partial^2 H}{\partial \xi \partial D} - b \theta \xi \frac{\partial^2 H}{\partial \xi \partial \theta} - b \sigma D \frac{\partial^2 H}{\partial \theta \partial D} \tag{C.2}
\]

with boundary condition \( H() = D_t^n\xi^m_t \). We conjecture that the solution is of the form

\[
H() = D_t^n\xi^m_t e^{n(\mu - \frac{1}{2}\sigma^2)\tau + \frac{\sigma^2}{2} \tau \xi^2} e^{A_0(\tau) + n A_1(\tau) + \sigma^2 A_2(\tau)}. \tag{C.3}
\]

Plugging Equation (C.3) into Equation (C.2) we find that \((A_0, A_1, A_2)\) is a solution to the system

\[
A_2' = a_2 A_2^2 - 2a_1 A_2 + a_0 \tag{C.4}
\]

\[
A_1' = (a_2 A_2 - a_1) A_1 + nm \sigma - 2nb \sigma A_2 \tag{C.5}
\]

\[
A_0' = \frac{1}{4} a_2 (A_1^2 + 2A_2) - nb \sigma A_1 \tag{C.6}
\]

with boundary conditions

\[
A_2(0) = 0; \quad A_1(0) = 0; \quad A_0(0) = 0,
\]
where
\[ a_0 = \frac{1}{2} m(m - 1); \quad a_1 = \kappa + mb; \quad a_2 = 2b^2. \]

Let
\[ q = \sqrt{a_1^2 - a_2a_0}. \]

The solution to Equation (C.20) is
\[ A_2(\tau) = \frac{a_0 (1 - e^{-2q\tau})}{q + a_1 + (q - a_1)e^{-2q\tau}}. \]  

(C.7)

We now turn our attention to Equation (C.21). One can verify that
\[ \int A_2(\tau) d\tau = \frac{a_1 - q}{a_2} \tau - \frac{1}{a_2} \ln \left[ q + a_1 + (q - a_1)e^{-2q\tau} \right] \]
\[ \Rightarrow - \int (a_2 A_2 - a_1) d\tau = q\tau + \ln \left[ q + a_1 + (q - a_1)e^{-2q\tau} \right]. \]

Therefore,
\[ u(\tau) \equiv e^{-\int a_2[A_2(\tau) + a_1] d\tau} = \left[ q + a_1 + (q - a_1)e^{-2q\tau} \right] e^{q\tau} \]
\[ \int u(\tau) d\tau = \frac{1}{q} \left[ q + a_1 - (q - a_1)e^{-2q\tau} \right] e^{q\tau}. \]

Moreover,
\[ u' = -(a_2 A_2 - a_1)u \quad \Rightarrow \quad \int u(\tau) A_2(\tau) = \frac{1}{a_2} \left[ a_1 \int u(\tau) d\tau - u(\tau) \right]. \]
Define

\[
H(\tau) \equiv \left[ mn \sigma \int u(\tau) d\tau - 2nb \sigma \int u(\tau) A_2(\tau) d\tau \right] \\
= \frac{n \sigma}{a_2} \left[ \frac{(ma_2 - 2ba_1)}{a_3} \int u(\tau) d\tau + 2bu(\tau) \right] \\
= \frac{n \sigma}{a_2q} \left\{ a_3 \left[ q + a_1 - (q - a_1)e^{-2q\tau} \right] + 2bq \left[ q + a_1 + (q - a_1)e^{-2q\tau} \right] \right\} e^{q\tau}
\]

Equation (C.21) is a First-Order ODE with \( u \) as integrating factor. Its solution is

\[
A_1(\tau) = \frac{1}{u(\tau)} \left[ H(\tau) - H(0) \right] \\
= \frac{n \sigma}{a_2q} \frac{a_3}{a_1} \left[ q + a_1 - (q - a_1)e^{-2q\tau} - 2a_1e^{-q\tau} \right] + 2bq \left[ q + a_1 + (q - a_1)e^{-2q\tau} - 2qe^{-q\tau} \right] \\
= \frac{n \sigma}{a_2q} \frac{q}{q + a_1 + (q - a_1)e^{-2q\tau}}
\]

where we used the boundary condition for \( A_1 \) in the first equality.

Finally, the solution to Equation (C.22) is

\[
A_0(\tau) = \sum_{i=0}^{2} d_i D_1(i, \tau) + \sum_{i=0}^{4} e_i D_2(i, \tau) \tag{C.9}
\]

where

\[
c_1 = c_0(a_3 + 2bq)(q + a_1); \quad c_2 = -2c_0(a_1a_3 + 2bq^2); \quad c_3 = c_0(q - a_1)(2bq - a_3); \quad c_0 = \frac{n \sigma}{qa_2};
\]
\[
d_0 = b^2a_0 - nb\sigma c_1; \quad d_1 = -nb\sigma c_2; \quad d_2 = -(b^2a_0 + nb\sigma c_3);
\]
\[
e_0 = \frac{b^2c_1^2}{2}; \quad e_1 = b^2c_1c_2; \quad e_2 = \frac{b^2(2c_1c_3 + c_2^2)}{2};
\]
\[
e_3 = b^2c_2c_3; \quad e_4 = \frac{b^2c_3^2}{2}; \quad a_3 = ma_2 - 2ba_1
\]
The functions $D_i(j, \tau)$ are

\[
D_1(j, \tau) = \int_0^\tau \frac{e^{-jqs}}{q + a_1 + (q - a_1)e^{-2qs}} ds
\]

\[
= \begin{cases} 
\frac{\tau}{q + a_1} + \frac{1}{2q(q + a_1)} \ln \frac{q + a_1 + (q - a_1)e^{-2q\tau}}{2q} & \text{if } j = 0 \\
\frac{1}{q(q + a_1)} \left[ 2F_1 \left( 1, \frac{j}{2}; \frac{j}{2} + 1; z \right) - e^{-j\tau} 2F_1 \left( 1, \frac{j}{2}; \frac{j}{2} + 1; \bar{z} \right) \right] & \text{if } j > 0.
\end{cases}
\]

\[
D_2(j, \tau) = \int_0^\tau \frac{e^{-jqs}}{[q + a_1 + (q - a_1)e^{-2qs}]^2} ds
\]

\[
= \frac{1}{2q(q - a_1)} \left[ \frac{e^{(2-j)q\tau}}{q + a_1 + (q - a_1)e^{-2q\tau}} - \frac{1}{2q} + (j - 2)qD_1(j - 2, \tau) \right].
\]

Here,

\[
z = -\frac{q - a_1}{q + a_1} \quad \text{and} \quad \bar{z} = e^{-2q\tau} z.
\]

The functions are well defined since

\[
|\bar{z}| \leq z < 1
\]

for the parameters we consider.
Suppose that \( j > 0 \).

\[
D_1(j, \tau) = \int_0^\tau \frac{e^{-jqs}}{q + a_1 + (q - a_1)e^{-2qs}} ds
\]
\[
= \frac{1}{q + a_1} \int_0^\tau e^{-jqs} (1 - ze^{-2qs})^{-1} ds
\]
\[
= \frac{1}{q + a_1} \int_0^\tau (e^{-2qs})^{\frac{1}{2} - 1} (1 - ze^{-2qs})^{-1} e^{-2qs} ds
\]
\[
= \frac{1}{-2q(q + a_1)} \int_1^1 x^{\frac{j}{2} - 1} (1 - zx)^{-1} dx
\]
\[
= \frac{1}{2q(q + a_1)} \int_{e^{-2q\tau}}^1 x^{\frac{j}{2} - 1} (1 - zx)^{-1} dx
\]
\[
= \frac{1}{2q(q + a_1)} \left[ \int_0^1 x^{\frac{j}{2} - 1} (1 - zx)^{-1} dx - \int_0^{e^{-2q\tau}} x^{\frac{j}{2} - 1} (1 - zx)^{-1} dx \right]
\]
\[
= \frac{1}{2q(q + a_1)} \left[ \int_0^1 x^{\frac{j}{2} - 1} (1 - zx)^{-1} dx - e^{-j \tau} \int_0^1 x^{\frac{j}{2} - 1} (1 - zx)^{-1} dx \right]
\]
\[
= \frac{1}{2q(q + a_1)} \left[ \frac{2F_1\left(1, \frac{j}{2}, \frac{j}{2} + 1; z\right) - e^{-j \tau} \frac{2F_1\left(1, \frac{j}{2}, \frac{j}{2} + 1; \tilde{z}\right)}}{j} \right]
\]
Appendix C.5. One Freddy, Two Trees

Let $P_\alpha$ be the price of an asset paying dividends

$$D_{j,u}^\alpha \equiv D_{1,u}^{\alpha_1}D_{2,u}^{\alpha_2}, \text{ where } \alpha = (\alpha_1, \alpha_2).$$

Let

$$D_\alpha^u \equiv D_u^\alpha, \text{ where } \alpha = \alpha_1 + \alpha_2.$$

Then,

$$\left[ (1 + \alpha^{1/\gamma} \xi_t^{1/\gamma})^\gamma D_t^{-\gamma} \right] P_\alpha = \int_0^\infty e^{-\beta(u-t)} E_t \left\{ (1 + \alpha^{1/\gamma} \xi_u^{1/\gamma})^\gamma D_u^{-\gamma} D_{j,u}^\alpha \right\} du.$$\hfill (C.10)

Computing this ratio will require computing expectations of the form

$$H() = E\left\{ \xi^m D_u^{-n} D_{1,u}^\alpha D_{2,u}^\beta \right\}.$$

We proceed as before:

$$H() = E\left[ \xi_u^m e^{\alpha y_1,u + \beta y_2,u} \right]$$

$$= \int_{-\infty}^{\infty} E\left[ \xi_u^m e^{(\alpha-n/2-iz)y_1,u + (\beta-n/2+iz)y_2,u} \right] F(z)dz$$

$$= \int_{-\infty}^{\infty} E\left[ e^{m \log \xi_u + (\alpha-n/2-iz)y_1,u + (\beta-n/2+iz)y_2,u} \right] F(z)dz. \hfill (C.10)$$
\[
H() = \mathbb{E}\left[ e^{m \log \xi_u (e^{\tilde{y}_{1,u} + \tilde{y}_{2,u}})} \right] \\
= e^{-\frac{\mu_1 \tau + \mu_2 \tau}{2}} \int_{-\infty}^{\infty} e^{i(y_{2,t} - y_{1,t})} \mathbb{E}\left[ e^{m e^{(\alpha - n/2 - iz)\tilde{y}_{1,u} + (\beta - n/2 + iz)\tilde{y}_{2,u}}} \right] F(z) dz \\
= e^{-\frac{\mu_1 \tau + \mu_2 \tau}{2}} \int_{-\infty}^{\infty} e^{i(y_{2,t} - y_{1,t})} \mathbb{E}\left[ e^{m \log \xi_u + (\alpha - n/2 - iz)\tilde{y}_{1,u} + (\beta - n/2 + iz)\tilde{y}_{2,u}} \right] F(z) dz.
\]

(C.11)

Consider the following expectation:

\[
G() = \mathbb{E}\left[ e^{m \log \xi_u + p_1 y_{1,u} + p_2 y_{2,u}} \right] \\
= \mathbb{E}\left[ e^{m (\log \xi_1,u + \log \xi_2,u) + p_1 y_{1,u} + p_2 y_{2,u}} \right] \\
= \mathbb{E}\left[ e^{m \log \xi_1,u + p_1 y_{1,u}} \right] \mathbb{E}\left[ e^{m \log \xi_2,u + p_2 y_{2,u}} \right]
\]

where the last equality follows from the fact that the Brownian motions are uncorrelated:

\[ dB_2 dB_2 = 0. \]

We use the result from Equation (C.3) to compute each of the two expectations above:

\[
\mathbb{E}\left[ e^{m \log \xi_j,u + p_j y_{j,u}} \right] = e^{m \log \xi_{2,t} + p_{2,y_{2,t}}} e^{c_j(p_j) \tau + A_{j,0}(\tau) + \theta_j A_{j,1}(\tau) + \theta^2_j A_{j,2}(\tau)}
\]

where

\[
c_j(x) = \mu_j x + \frac{x(x - 1)}{2} \sigma_j^2
\]

and \((A_{j,0}, A_{j,1}, A_{j,2})\) is obtained using Equations (C.7)–(C.9).
We now use the solution for the function $G$ in the Equation (C.11):

$$H() = e^{-n \frac{y_{1,t} + y_{2,t}}{2}} \int_{-\infty}^{\infty} e^{i[y_{2,t} - y_{1,t}]z} e^{m \log \xi_t + \tau c(\alpha - n/2 - iz; \beta - n/2 + iz)} e^{K(\theta_1, \theta_2, \tau)} F(z) dz$$

(C.12)

where

$$K(\theta_1, \theta_2, \tau) = \sum_{j=1}^{2} \left[ A_{j,0}(\tau) + \theta_j A_{j,1}(\tau) + \theta_j^2 A_{j,2}(\tau) \right].$$
Appendix C.6. One Freddy, N Trees

Let $P_\alpha$ be the price of an asset paying dividends

\[ D_j^\alpha \equiv \prod D_j^{\alpha_j}, \quad \text{where} \quad \alpha = (\alpha_1, \cdots, \alpha_N). \]

Let

\[ D^\alpha \equiv D^\alpha, \quad \text{where} \quad \alpha = \sum \alpha_j. \]

Then,

\[ \frac{P_\alpha}{D_j^\alpha} = E_t \left\{ \int_0^\infty e^{-\beta(u-t)} \left( \frac{1 + \alpha^{1/\gamma} \xi^{1/\gamma}}{1 + \alpha^{1/\gamma} \xi^{1/\gamma}} \right)^\gamma \left( \frac{D_u}{D_t} \right)^{-\gamma} \frac{D_j^{\alpha_j} u}{D_j^{\alpha_j, t}} du \right\}. \]

Computing this ratio will require computing expectations of the form

\[ H() = E \left\{ \left( \frac{1 + \alpha^{1/\gamma} \xi^{1/\gamma}}{1 + \alpha^{1/\gamma} \xi^{1/\gamma}} \right)^m \left( \frac{D_u}{D_t} \right)^n \prod \left( \frac{D_j^{\alpha_j} u}{D_j^{\alpha_j, t}} \right)^{\alpha_j} \right\} \]  

\[ H() = E \left\{ \xi^m D_u^{-n} \prod \left( \frac{D_j^{\alpha_j} u}{D_j^{\alpha_j, t}} \right)^{\alpha_j} \right\}. \]  

\[ \text{(C.13)} \]

We proceed as before:

\[ H() = E \left\{ \xi^m \frac{e^{\alpha' y_u}}{\sum e^{y_{j_u} + y_{j_t}}} \right\} \]

\[ = e^{-n'y_{t}/N} \int e^{iz'(Qy_{t})} E \left\{ \xi^m (\alpha - n/N + iQ' z') y_{u} \right\} F(z) dz \]

\[ = e^{-n'y_{t}/N} \int e^{iz'(Qy_{t})} E \left\{ e^{m \log (\alpha - n/N + iQ' z') y_{u}} \right\} F(z) dz \]

\[ = e^{-n'y_{t}/N} \int e^{iz'(Qy_{t})} e^{m \log \xi + \alpha - n/N + iQ' z'} y_{u} K_N(\theta_1, \theta_2, \tau) F_N(z) dz \]
where

\[ K_N(\theta, \tau) = \sum_{j=1}^{N} \left[ A_{j,0}(\tau) + \theta_j A_{j,1}(\tau) + \theta_j^2 A_{j,2}(\tau) \right]. \]
Appendix C.7. Two Freddy, One Tree

Let $H$ be the expectation

$$H(\tau, D_t, \xi_t, x, y; n, m) = E[\eta_{1,u}^{n} D_u^{m} \xi_u] , \quad (C.14)$$

where

$$x = \theta_{1,t} \text{ and } y = \theta_{2,t}.$$ 

By the Feynman-Kac formula, $H$ satisfies the PDE

$$0 = -\frac{\partial H}{\partial \tau} + \mu D \frac{\partial H}{\partial D} + \mu_x \frac{\partial H}{\partial x} + \mu_y \frac{\partial H}{\partial y} - x(y - x)\xi \frac{\partial H}{\partial \xi}$$

$$+ \frac{1}{2} (\sigma_D)^2 \frac{\partial^2 H}{\partial D^2} + \frac{1}{2} \sigma_x^2 \frac{\partial^2 H}{\partial x^2} + \frac{1}{2} \sigma_y^2 \frac{\partial^2 H}{\partial y^2} + \frac{1}{2} (y - x)^2 \xi^2 \frac{\partial^2 H}{\partial \xi^2} + \frac{1}{2} \sigma^2 \eta_1^2 \frac{\partial^2 H}{\partial \eta_1^2}$$

$$+ (y - x)\xi \left[ \sigma D \frac{\partial H}{\partial D} \frac{\partial H}{\partial \xi} + \sigma_x \frac{\partial H}{\partial x} \frac{\partial H}{\partial \xi} + \sigma_y \frac{\partial H}{\partial y} \frac{\partial H}{\partial \xi} + x\eta_1 \frac{\partial H}{\partial \eta_1} \frac{\partial H}{\partial \xi} \right]$$

$$+ \sigma D \left[ \sigma_x \frac{\partial H}{\partial x} \frac{\partial H}{\partial D} + \sigma_y \frac{\partial H}{\partial y} \frac{\partial H}{\partial D} + x\eta_1 \frac{\partial H}{\partial D} \frac{\partial H}{\partial D} \right]$$

$$+ \theta_1 \eta_1 \left[ \sigma_x \frac{\partial H}{\partial x} \frac{\partial H}{\partial \eta_1} + \sigma_y \frac{\partial H}{\partial y} \frac{\partial H}{\partial \eta_1} \right] + \sigma_x \sigma_y \frac{\partial H^2}{\partial x \partial y} \quad (C.15)$$

with boundary condition

$$H(t, D_t, \xi_t, x, y; n, m) = \eta_{1,t}^{n} D_t^{m} \xi_t.$$ 

We conjecture that the solution is of the form

$$H(\tau) = \eta_{1,t}^{n} D_t^{m} \xi_t \exp \left\{ n \left( \mu - \frac{1}{2} \sigma^2 \right) \tau + \frac{n^2 \sigma^2}{2} \tau \right\} e^{h(x,y,\tau)} \quad (C.16)$$
\[ h(x, y, \tau) = A_0(\tau) + B_1(\tau)x + B_2(\tau)y + C_0(\tau)x^2 + 2C_1xy + B_2(\tau)y^2. \]

Plugging Equation (C.31) into Equation (C.30) we obtain

\[
\begin{align*}
\frac{\partial h}{\partial \tau} &= \mu_x \frac{\partial h}{\partial x} + \mu_y \frac{\partial h}{\partial y} - mx(y - x) + 1 \frac{\sigma_x^2}{2} \left( \left( \frac{\partial h}{\partial x} \right)^2 + \frac{\partial^2 h}{\partial x^2} \right) + 1 \frac{\sigma_y^2}{2} \left( \left( \frac{\partial h}{\partial y} \right)^2 + \frac{\partial^2 h}{\partial y^2} \right) \\
&\quad + \frac{m(m - 1)}{2} (y - x)^2 + m(y - x) \left[ \sigma_n + \sigma_x \frac{\partial h}{\partial x} + \sigma_y \frac{\partial h}{\partial y} \right] + n \sigma \left[ \sigma_x \frac{\partial h}{\partial x} + \sigma_y \frac{\partial h}{\partial y} \right] + x \left[ \sigma_x \frac{\partial h}{\partial x} + \sigma_y \frac{\partial h}{\partial y} \right] + \sigma_x \sigma_y \left[ \frac{\partial h}{\partial x} \frac{\partial h}{\partial y} + \frac{\partial^2 h}{\partial x \partial y} \right] \\
&\quad + 1 \frac{\sigma_x^2}{2} \left( \left( \frac{\partial h}{\partial x} \right)^2 + \frac{\partial^2 h}{\partial x^2} \right) + 1 \frac{\sigma_y^2}{2} \left( \left( \frac{\partial h}{\partial y} \right)^2 + \frac{\partial^2 h}{\partial y^2} \right) + \sigma_x \sigma_y \left[ \frac{\partial h}{\partial x} \frac{\partial h}{\partial y} + \frac{\partial^2 h}{\partial x \partial y} \right].
\end{align*}
\]

We assume that
\[ \mu_x = \kappa_x x; \quad \mu_y = \kappa_y y \]
and that $\kappa_x, \kappa_y, \sigma_x, \sigma_y$ are all constant. We define

$$X = \begin{bmatrix} x \\ y \end{bmatrix}; \quad S = \begin{bmatrix} \sigma_x \\ \sigma_y \end{bmatrix}; \quad \frac{\partial h}{\partial X} = \begin{bmatrix} \frac{\partial h}{\partial x} \\ \frac{\partial h}{\partial y} \end{bmatrix}; \quad \frac{\partial^2 h}{\partial X \partial X'} = \begin{bmatrix} \frac{\partial^2 h}{\partial x^2} & \frac{\partial^2 h}{\partial x \partial y} \\ \frac{\partial^2 h}{\partial y \partial x} & \frac{\partial^2 h}{\partial y^2} \end{bmatrix}$$

$$D = \begin{bmatrix} \kappa_x + (1 - m)\sigma_x & m\sigma_x \\ (1 - m)\sigma_y & \kappa_y + m\sigma_y \end{bmatrix}; \quad E = \frac{m(m - 1)}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}; \quad V = n\sigma \begin{bmatrix} 1 - m \\ m \end{bmatrix};$$

$$\mu_X = n\sigma S + DX$$

$$h = A + B'X + X'CX.$$ 

We can rewrite the Feynman-Kac equation as

$$\frac{\partial h}{\partial \tau} = \frac{m(m - 1)}{2} (y - x)^2 + mn\sigma(y - x) + n\sigma x + \mu'_X \frac{\partial h}{\partial X} + \frac{1}{2} \left[ S' \frac{\partial^2 h}{\partial X \partial X'} S + \left( S' \frac{\partial h}{\partial X} \right)^2 \right]$$

$$= X'EX + V'X + \mu'_X [B + 2CX] + \frac{1}{2} \left[ 2S'C'S + (S'B + 2S'CX)^2 \right]$$

$$= S'C'S + n\sigma B + \frac{1}{2} (S'B)^2 + \left[ V' + B'D' + 2n\sigma S'C + 2B'SS'C \right] X$$

$$+ X' \left[ CD + D'C + 2C'SSS'C + E \right] X.$$ 

We used the following relation in the last equality:

$$2X'D'CX = X'D'CX + X'D'CX$$

$$= X'D'CX + (X'D'CX)'$$

$$= X'D'CX + X'C'DX$$

$$= X'D'CX + X'CDX$$

$$= X' [D'C + CD] X$$
where the second equality follows from the fact that $X'D'CX$ is a real number and the fourth equality follows from the fact that $C$ is symmetric. We perform this transformation to obtain a symmetric term $(D'C + CD)$ instead of working with a non-symmetric term $(D'C)$ because $C$ is symmetric. The importance of this transformation will become clear later on.

We deduce that $(A, B, C)$ is a solution to the following system:

$$\frac{\partial C}{\partial \tau} = 2C'SS'C + CD + D'C + E \quad \text{(C.17)}$$
$$\frac{\partial B}{\partial \tau} = [D + 2C'SS']B + V + 2n\sigma C'S \quad \text{(C.18)}$$
$$\frac{\partial A}{\partial \tau} = S'CS + n\sigma S'B + \frac{1}{2}(S'B)^2. \quad \text{(C.19)}$$

A necessary condition for Equation (C.17) to have a solution is that its RHS is symmetric because its LHS is symmetric since it is the case for $C$. This condition holds because both $E$ and $D'C + CD$ are symmetric.

**Remark:** We recover the “One Freddy” case by setting

$$x = (\kappa_x = ) \sigma_x = 0.$$

$$A'_2 = a_2 A_2^2 - 2a_1 A_2 + a_0 \quad \text{(C.20)}$$
$$A'_1 = (a_2 A_2 - a_1)A_1 + n\mu\sigma - 2n\sigma A_2 \quad \text{(C.21)}$$
$$A'_0 = \frac{1}{4}a_2(A_1^2 + 2A_2) - n\sigma A_1 \quad \text{(C.22)}$$

Equation (C.17) is a matrix Riccati equation. Its solution is

$$C(\tau) = C_{22}^{-1}(\tau)C_{21}(\tau) \quad \text{(C.23)}$$
where
\[
\begin{bmatrix}
C_{11}(\tau) & C_{21}(\tau) \\
C_{12}(\tau) & C_{22}(\tau)
\end{bmatrix}
= \exp \left\{ \tau \begin{bmatrix}
D & -2SS' \\
E & -D
\end{bmatrix} \right\}.
\]

Consider Equation (C.18). The integrating factor of this equation is
\[
H(\tau) = e^{\int (D + 2C'SS') d\tau}.
\]
The solution to Equation (C.18) is thus
\[
B(\tau) = H^{-1}(\tau) \int_0^\tau H(s) \left[ V + 2n\sigma C'S \right] ds.
\]
(C.24)

The solution to Equation (C.19) is obtained through an integral once we have both \(C\) and \(B\):
\[
A(\tau) = \int_0^\tau \left[ S'C(s)S + n\sigma S'B(s) + \frac{1}{2} (S'B(s))^2 \right] ds.
\]
(C.25)

Note that the integrand in the RHS of Equation (C.25) is a real-value function. Thus, the integral is simple to evaluate numerically.

The matrix exponential are difficult to compute for arbitrary parameter values. For a given set of values, we shall use the Jordan decomposition to compute the matrix exponentials.

**Lemma 1.** Suppose that
\[
\kappa_x = \kappa_y = \kappa \quad \text{and} \quad \sigma_x = -\sigma_y.
\]
Then, the $4 \times 4$ matrix

$$
\begin{bmatrix}
D & -2SS' \\
E & -D
\end{bmatrix}
$$

has at least two real eigenvalues, one of which is

$\kappa$.

We can use the lemma above when finding a simple form for $C$. 
Appendix C.8. Two Freddy, Two Trees

\[ H() = E \left\{ \eta_{1,u} \xi^m D_u^{-n} \left( \frac{D_{1,u}}{D_{1,t}} \right)^\alpha \left( \frac{D_{2,u}}{D_{2,t}} \right)^\beta \right\}. \]

We proceed as before:

\[
H() = E \left[ \eta_{1,u} \xi^m \left( e^{(\alpha y_{1,u} + \beta y_{2,u})} \right) \right] \\
= e^{-\frac{y_{1,t} + y_{2,t}}{2}} \int_{-\infty}^{\infty} e^{i[y_{2,t}-y_{1,t}]z} \left[ \eta_{1,u} \xi^m e^{(\alpha-n/2-iz)y_{1,u} + (\beta-n/2+iz)y_{2,u}} \right] F(z) dz \\
= e^{-\frac{y_{1,t} + y_{2,t}}{2}} \int_{-\infty}^{\infty} e^{i[y_{2,t}-y_{1,t}]z} \eta_{1,t} \xi^m e^{(\alpha-n/2-iz;\beta-n/2+iz)} e^{K(\theta_1, \theta_2, \tau)} F(z) dz
\]

where

\[
K(\theta_1, \theta_2, \tau) = \sum_{j=1}^{2} \left[ A_{j,0}(\tau) + \theta'_j B_j(\tau) + \theta'_j C_{j,2}(\tau) \theta_j \right]
\]

where

\[ \theta_j = (\theta_{1,j}, \theta_{2,j})'. \]
Appendix C.9. Two Freddy, $N$ Trees

$$H() = E \left\{ \xi^m D_u^{-n} \prod \left( \frac{D_{j,u}}{D_{j,t}} \right)^{\alpha_j} \right\}.$$

We proceed as before:

$$H() = E \left\{ \xi^m \frac{e^{\alpha' \tilde{y}_u}}{\left( \sum e^{\tilde{y}_{j,u} + \tilde{y}_{j,t}} \right)^{\alpha}} \right\}$$

$$= e^{-n' y_t / N} \int e^{iz'(Q y_t) E \left\{ \xi^m e^{(\alpha - n/N + iQ' z) \tilde{y}_u} \right\}} F(z) \, dz$$

$$= e^{-n' y_t / N} \int e^{iz'(Q y_t) E \left\{ e^{m \log \xi + (\alpha - n/N + iQ' z) \tilde{y}_u} \right\}} F(z) \, dz$$

$$= e^{-n' y_t / N} \int e^{iz'(Q y_t)} e^{m \log \xi + \tau c((\alpha - n/N + iQ' z) \tilde{y}_u)} e^{K_N(\theta_1, \theta_2, \tau)} F_N(z) \, dz$$

where

$$K_N(\theta, \tau) = \sum_{j=1}^N \left[ A_{j,0}(\tau) + \theta_j A_{j,1}(\tau) + \theta_j^2 A_{j,2}(\tau) \right].$$
\begin{align*}
\frac{\partial h}{\partial \tau} &= -mxy + \frac{1}{2} m(m-1)y^2 + y[mn\sigma + mx] + n\sigma x \\
&\quad + \mu_x h_x + \mu_y h_y + \frac{1}{2} \sigma_x^2 (h_x^2 + h_{xx}) + \frac{1}{2} \sigma_y^2 (h_y^2 + h_{yy}) \\
&\quad + my [\sigma_x h_x + \sigma_y h_y] + n\sigma [\sigma_x h_x + \sigma_y h_y] \\
&\quad + x [\sigma_x h_x + \sigma_y h_y] + \sigma_x \sigma_y (h_x h_y + h_{xy})
\end{align*}

where

\begin{align*}
h(\tau, x, y) &\equiv A_0(\tau) + A_1(\tau)x + B_0(\tau) + B_1(\tau)y + B_2(\tau)y^2.
\end{align*}

\begin{align*}
\frac{\partial h}{\partial \tau} &= \frac{1}{2} m(m-1)y^2 + mn\sigma y + n\sigma x + \sigma_x \sigma_y (h_x h_y + h_{xy}) \\
&\quad + \mu_x h_x + \mu_y h_y + \frac{1}{2} \sigma_x^2 (h_x^2 + h_{xx}) + \frac{1}{2} \sigma_y^2 (h_y^2 + h_{yy}) \\
&\quad + (my + x) [\sigma_x h_x + \sigma_y h_y] + n\sigma [\sigma_x h_x + \sigma_y h_y]
\end{align*}

A necessary condition for this approach to work is that the “second” part of the RHS above is polynomial. One way to achieve it is to model $\mu_{\theta_1}, \mu_{\theta_2}$, and $h$ as polynomial functions.

**Assumptions:**

\[ \mu_{\theta_1} = a_I + b_I \theta_I \quad \text{and} \quad \sigma_{\theta_I} \text{ is constant.} \]

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\[ \mu_{\theta_1} = a_I + b_I \theta_I \quad \text{and} \quad \sigma_{\theta_I} \text{ is constant.} \]
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\frac{\partial h}{\partial \tau} = -m\theta_1(\theta_2 - \theta_1) + \frac{1}{2} m(m - 1)(\theta_2 - \theta_1)^2 + (\theta_2 - \theta_1)m\theta_1[n\sigma + m\theta_1] + n\sigma \theta_1 + \\
\mu_\theta_1 h_1 + \mu_\theta_2 h_2 + \frac{1}{2}\sigma_{\theta_1}^2(h_1^2 + h_{11}) + \frac{1}{2}\sigma_{\theta_2}^2(h_2^2 + h_{22}) + (\theta_2 - \theta_1)[m\sigma\theta_1 h_1 + m\sigma\theta_2 h_2] + \\
\sigma [n\sigma\theta_1 h_1 + n\sigma\theta_2 h_2] + \theta_1 [\sigma\theta_1 h_1 + \sigma\theta_2 h_2] + \sigma\theta_1 \sigma\theta_2 (h_1 h_2 + h_{12})
\]

Plugging Equation (C.31) into Equation (C.30) we find that \((A_0, A_1, A_2)\) is a solution to the system

\[
\begin{align*}
A'_2 &= a_2 A_2^2 - 2a_1 A_2 + a_0 \quad \text{(C.26)} \\
A'_1 &= (a_2 A_2 - a_1)A_1 + nm\sigma \quad \text{(C.27)} \\
A'_0 &= \frac{1}{4} a_2 (A_1^2 + 2A_2) - \frac{na_2}{2}\sigma(A_1 + 2A_2) \quad \text{(C.28)}
\end{align*}
\]

with boundary conditions

\[
A_2(0) = 0; \quad A_1(0) = 0; \quad A_0(0) = 0,
\]

where

\[
a_0 = \frac{1}{2} m(m - 1); \quad a_1 = \kappa + mb; \quad a_2 = 2b^2.
\]

Let

\[
q = \sqrt{a_1^2 - a_2 a_0}.
\]
Let $H$ be the expectation

$$H(\tau, D_t, \xi_t, x, y; n, m) = E[\eta_{1,u} D_u^n \xi_u^m], \quad (C.29)$$

where

$$x = \theta_{1,t} \quad \text{and} \quad y = \theta_{2,t} - \theta_{1,t}.$$ 

By the Feynman-Kac formula, $H$ satisfies the PDE

$$0 = -\frac{\partial H}{\partial \tau} + \mu D \frac{\partial H}{\partial D} + \mu_x \frac{\partial H}{\partial x} + \mu_y \frac{\partial H}{\partial y} - xy\xi \frac{\partial H}{\partial \xi}
+ \frac{1}{2} (\sigma D)^2 \frac{\partial^2 H}{\partial D^2} + \frac{1}{2} \sigma_x^2 \frac{\partial^2 H}{\partial x^2} + \frac{1}{2} \sigma_y^2 \frac{\partial^2 H}{\partial y^2} + \frac{1}{2} \sigma_1^2 \eta_1^2 \frac{\partial^2 H}{\partial \eta_1^2}
+ y\xi \left[ \sigma_D \frac{\partial H^2}{\partial D \partial \xi} + \sigma_x \frac{\partial H^2}{\partial x \partial \xi} + \sigma_y \frac{\partial H^2}{\partial y \partial \xi} + \theta_1 \eta_1 \frac{\partial H^2}{\partial \eta_1 \partial \xi} \right]
+ \sigma D \left[ \sigma_x \frac{\partial H^2}{\partial x \partial D} + \sigma_y \frac{\partial H^2}{\partial y \partial D} + \theta_1 \eta_1 \frac{\partial H^2}{\partial \eta_1 \partial D} \right]
+ \theta_1 \eta_1 \left[ \sigma_x \frac{\partial H^2}{\partial x \partial \eta_1} + \sigma_y \frac{\partial H^2}{\partial y \partial \eta_1} \right] + \sigma_x \sigma_y \frac{\partial H^2}{\partial x \partial y} \quad (C.30)$$

with boundary condition

$$H(t, D_t, \xi_t, x, y; n, m) = \eta_{1,t} D_t^n \xi_t^m.$$ 

We conjecture that the solution is of the form

$$H() = \eta_{1,t} D_t^n \xi_t^m \exp \left\{ n \left( \mu - \frac{1}{2} \sigma^2 \right) \tau + \frac{n^2}{2\sigma^2} \tau^2 \right\} e^{h(x,y,\tau)} \quad (C.31)$$

where

$$h(x, y, \tau) = A_0(\tau) + B_1(\tau)x + B_2(\tau)y + C_0(\tau)x^2 + 2C_1xy + B_2(\tau)y^2.$$
\[
\frac{\partial h}{\partial \tau} = -mxy + \frac{1}{2}m(m-1)y^2 + y\left[mn\sigma + mx\right] + n\sigma x \\
+ \mu_x h_x + \mu_y h_y + \frac{1}{2}\sigma_x^2(h_x^2 + h_{xx}) + \frac{1}{2}\sigma_y^2(h_y^2 + h_{yy}) \\
+ my[\sigma_x h_x + \sigma_y h_y] + n\sigma [\sigma_x h_x + \sigma_y h_y] \\
+ x[\sigma_x h_x + \sigma_y h_y] + \sigma_x \sigma_y (h_x h_y + h_{xy})
\]

where

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h(\tau, x, y) \equiv A_0(\tau) + A_1(\tau)x + B_0(\tau) + B_1(\tau)y + B_2(\tau)y^2.
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+ (my + x)[\sigma_x h_x + \sigma_y h_y] + n\sigma [\sigma_x h_x + \sigma_y h_y]
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A necessary condition for this approach to work is that the “second” part of the RHS above is polynomial. One way to achieve it is to model \(\mu_\theta_1, \mu_\theta_2\), and \(h\) as polynomial functions. aaa

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\]
\[
+ \mu_0_1 h_1 + \mu_0_2 h_2 + \frac{1}{2}\sigma_0_1^2(h_1^2 + h_{11}) + \frac{1}{2}\sigma_0_2^2(h_2^2 + h_{22})
\]
\[
+ (\theta_2 - \theta_1)[m\sigma_\theta_1 h_1 + m\sigma_\theta_2 h_2]
\]
\[
+ \sigma[n\sigma_\theta_1 h_1 + n\sigma_\theta_2 h_2]
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+ \theta_1[\sigma_\theta_1 h_1 + \sigma_\theta_2 h_2] + \sigma_\theta_1 \sigma_\theta_2 (h_1 h_2 + h_{12})
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Plugging Equation (C.31) into Equation (C.30) we find that \((A_0, A_1, A_2)\) is a solution to the system

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A_0' = \frac{1}{4}a_2(A_1^2 + 2A_2) - \frac{na_2^2}{2}\sigma(A_1 + 2A_2) \quad \text{(C.34)}
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a_0 = \frac{1}{2}m(m-1); \quad a_1 = \kappa + mb; \quad a_2 = 2b^2.
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Let

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q = \sqrt{a_1^2 - a_2 a_0}.
\]
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