Chapter 12 — Cumulative Prospect Theory

After Epstein-Zin Preferences, Cumulative Prospect Theory is probably the most commonly used alternative to Expected Utility Theory. CPT differs from EUT in a number of important ways. In place of probabilities, it uses probability weights of the type described in the previous chapter. It evaluates not the level of consumption or wealth, but deviations in it from some reference level. Finally, the utility or value function is not concave, but S-shaped. It is convex below the reference level and concave above it. Choices are evaluated as

$$E_{\omega}[v(\bar{x})] = \sum \omega_x(\pi_x)v(x_x)$$  \hspace{1cm} (1)

where $v$ is the utility function and $\omega$ are the probability weights. The latter depend on all of the outcomes and the entire probability distribution just as in RDU.

By convention in CPT, outcomes are first classified as gains and losses relative to the reference level. If consumption is the relevant object, and the reference level is $c^o$, then every outcome below $c^o$ is termed a loss and represented as a negative number $c_x - c^o \equiv x_x < 0$. Similarly, outcomes above $c^o$ are termed gains and represented as a positive number $c_x - c^o \equiv x_x > 0$. The $n$ losses and $m$ gains are ordered so that

$$x_{-n} \leq x_{-n+1} \leq \cdots \leq x_{-1} \leq x_0 = 0 \leq x_1 \leq \cdots \leq x_m.$$  \hspace{1cm} (2)

Again by convention the utility of the reference level is 0, $v(x_0) \equiv v(0) = 0$.

Probability Weighing in Cumulative Prospect Theory

The probability weighting used in CPT has one important difference from that used in RDU. In CPT, the weights are determined separately for gains above the reference level and losses below it. Then the cumulative probabilities for the ranked losses and the complementary cumulative probabilities for the ranked gains are determined

$$P_i = p_{-n} + \cdots + p_{-i} \quad P_i^* = p_i + \cdots + p_m.$$  \hspace{1cm} (3)

Two weighting functions, $\Omega^\pm$, are applied separately to the cumulative and complementary cumulative probabilities, and the decision weights are determined by differencing

$$\omega_{-i} = \Omega^-(P_{i-1}) - \Omega^-(P_{i-1}) \quad \omega_i = \Omega^+(P_i) - \Omega^+(P_{i+1}) \quad \text{for } i > 0.$$  \hspace{1cm} (4)

For a continuous distribution with an objective differentiable cumulative distribution $P(s)$, the decision-weight density functions are

$$\omega^-(s) = -\frac{d\Omega^-(P(s))}{ds} = \frac{d\Omega^-(1-P(s))}{ds} = \frac{d\Omega^+(1-P(s))}{ds} = \frac{d\Omega^+(P(s))}{ds} = \pi(s)$$  \hspace{1cm} (5)

where $p(s)$ is the objective probability density function over the states. As under RDU, the weighting functions must be strictly increasing, and impossible and certain events must be mapped in an obvious fashion, $\Omega^\pm(0) = 0$ and $\Omega^\pm(1) = 1$. Tversky and Kahneman proposed the weighting function defined in the previous chapter.

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1 If two outcomes have the same value they can be merged into a single value, but this is not necessary. It is more convenient to keep them separate if the indices denote states, which could well have the same consumption. The total probability weight assigned to the states with the same value of $x$ is invariant to grouping. If $x_i = x_{i+1}$, then $\omega_{i+1} + \omega_i = \Omega(P_{i+1}) - \Omega(P_i) + \Omega(P_i) - \Omega(P_{i+1}) = \Omega(P_{i+1}) - \Omega(P_{i+1})$ which is exactly the same probability weight that would be assigned to the merged $i$ and $i+1$ states.
\[ \Omega(P) = \frac{P^\delta}{[P^\delta + (1 - P)^\delta]^1/\delta} \quad 0 < \delta \leq 1. \quad (6) \]

estimating the parameters \( \delta_- = 0.69 \) and \( \delta_+ = 0.61 \).

In a given application, there may or may not be an actual outcome at the reference level \( x_0 = 0 \). This is irrelevant for determining “expected” utility because \( v(0) = 0 \). However, in many contexts such as equilibrium models, it is the expectation of and covariance with marginal utility that is important so the numerical value of the associated probability weight, \( \omega_0 \), is required. Either \( \Omega^- \) and \( \Omega^+ \) could be extended to include the zero outcome; however, the two extension do not in general give the same value. Another obvious choice is to set \( \omega_0 = 1 - \Omega^-(P^-_1) - \Omega^+(P^+_1) \). This makes the sum of all the probability weights equal to one. Unfortunately, this choice is not always practical. First there may be no outcome \( x_0 \) to assign this weight. Second, this choice can be negative. Even when this residual assignment for \( \omega_0 \) is positive, it may differ markedly from the probability weights for similar outcomes. For example, for any risky gamble whose zero-gain threshold is in the range 40\% to 60\%, TK’s estimated decision weight functions with \( \delta^- = 0.69 \) and \( \delta^+ = 0.61 \) assigns total probability weights of less than 89\%. This leaves at least an 11\% assignment to \( x_0 \) regardless of how many outcomes there are. For a continuous distribution this would be an atom of at least 11\%. In many models with numerous outcomes, this would be unreasonably large. For example, suppose there is a 44\% chance of a loss and a 45\% chance of a gain with a 1\% of an outcome right at the reference level. The TK weighing functions would give weights of 41.7\% for all the losses, 39.5\% for all the gains, and an 18.8\% chance for the outcome at the reference level.

The problems of a negative or unreasonably large probability weight for the reference level can be avoided and a unique and reasonable probability weight for the reference outcome is assured for all gambles if the weighting functions satisfy \( \Omega^-(P) + \Omega^+ (1 - P) = 1 \quad \forall P \). In fact, this is both necessary and sufficient.

**Theorem 12.1: Unique Zero-Gain Decision Weight Assignment.** The probability weighting functions, \( \Omega^\pm \), both assign the same weight to the zero-change outcome for all gambles if and only if they satisfy \( \Omega^-(P) + \Omega^+ (1 - P) = 1 \quad \forall P \). This restriction is equivalent to using the single weighting function \( \Omega(P) \equiv \Omega^-(P) \) on the entire cumulative distribution.

**Proof:** Denote by \( P^-_1, P^+_1 \), and \( p_0 \), the cumulative probability of the smallest loss, the complementary cumulative probability of the smallest gain, and the probability of a gain of zero. The weights that would be applied to the zero gain from extending the loss and gain weighting functions are \( \omega^-_0 = \Omega^-(P^-_1 + p_0) - \Omega^-(P^-_1) \) and \( \omega^+_0 = \Omega^+(P^+_1 + p_0) - \Omega^+(P^+_1) \). If they are equal, then

\[ \Omega^- (1 - P^-_1) + \Omega^+ (P^+_1) = \Omega^- (1 - P^-_1) + \Omega^+ (P^-_1). \quad (7) \]

The left-hand side depends only on \( P^-_1 \) while the right-hand side depends only on \( P^-_1 \). Because (7) must apply to all risky prospects, \( P^-_1 \) and \( P^-_1 \) are arbitrary; therefore, both sides of the equation must be constant. This verifies that \( \Omega^-(P) + \Omega^+ (1 - P) \) is constant for all \( P \). Furthermore, as \( \Omega^\pm(0) = 0 \) and \( \Omega^\pm(1) = 1 \), this constant must be one. The converse is also obviously true.

Now identify \( \Omega(P) \equiv \Omega^-(P) \) as the single weighting function to be applied to the entire cumulative probability distribution. \( \Omega \) obviously assigns the same probability weights as \( \Omega^- \) to all losses. For gains, it assigns \( \omega^+_0 = \Omega^+(P^-_1) - \Omega^+(P^-_1) = 1 - \Omega^+ (1 - P^-_1) - 1 + \Omega^+(1 - P^-_1) \) which is the same weight that \( \Omega^+ \) assigns using the complementary cumulative probabilities.

As shown there are a number of advantages of assuming a single weighting function. Doing so assigns a weight to the reference outcome allowing for the computation of expected
marginal utility. The assigned weight is unique. It is not an atom when the underlying state space is continuous with no atoms in the natural distribution, and for discrete distributions, the weight is reasonable. From the figure in the previous chapter, it is clear that the weighting function is basically, though certainly not exactly, symmetric so there is little loss in fit in using a single weighting function.

**CPT and S-Shaped Utility**

The second main feature of CPT is the S-shaped value or utility function. It is based on gains and losses as described earlier. The function is strictly increasing and centered around the reference level which is set at zero with \( v(0) = 0 \). Recall that this is just a permissible affine transformation. It is concave for gains and convex for losses. The reference level is discussed in more detail later. If \( v \) is differentiable except possibly at zero, then\(^2\)

\[
\begin{align*}
\text{a) } & v'(x) > 0 & \text{← more is preferred to less} \\
\text{b) } & v''(x) \leq 0, x > 0 & \text{← risk aversion over gains} \\
\text{c) } & v''(x) \geq 0, x < 0 & \text{← risk preference over losses.} \\
\text{d) } & v'(0^+) / v'(0^-) \equiv \lambda \geq 1 & \text{← loss aversion}
\end{align*}
\]

Positive marginal utility, (a), is standard for differentiable utility functions. It means, for example, that first-degree stochastic dominance is still applicable. It is the last three properties, that distinguish a CPT utility function from the more usual utility function. Properties (b) and (c) together are referred to as an S-shape. Property (d) is often defined as (strict) loss aversion when \( \lambda > 1 \); however, this notion is discussed further below.

Even when utility is otherwise differentiable, it is often only continuous and explicitly not differentiable at 0, possessing only left and right derivative there with the latter smaller. That is, the utility function is often assumed to have something like first-order risk aversion even though it is not risk averse at all for negative outcomes.

Tversky and Kahneman (1992, henceforth TK) proposed and estimated a specific S-shaped utility function of the form

\[
v(x) = \begin{cases} 
x^\alpha & x \geq 0 \\
-\lambda |x|^\beta & x < 0
\end{cases}
\]

with \( 0 < \alpha, \beta \leq 1 \) and \( \lambda \geq 1 \). \(^{(9)}\)

Their estimated parameters are \( \alpha = 0.89 \) and \( \lambda = 2.25 \). Abdellaoui (2000) obtained the similar estimates \( \alpha = 0.89 \) and \( \beta = 0.92 \). These values of \( \alpha \) would seem to indicate only very mild aversion to risk because the relative risk aversion for gains for this function is \( 1 - \alpha \), and of course there is risk preference for losses. However, the parameter \( \lambda \) also strongly affects the aversion to risk. For example, using TK’s parameter values, a fifty-fifty gamble winning $1 or $2 has a certainty equivalent of $1.49 — one cent below the expected value, but an even chance at winning or losing a dollar has a certainty equivalent of $23¢. So while risk premiums over gambles with only gains are small, they are much larger when both gains and losses are involved.

An often used special case of TK utility is \( \alpha = \beta = 1 \). This is simply piecewise linear utility with a kink at \( x = 0 \).\(^3\) Although this piecewise linear function is technically S-shaped, it is also weakly concave (and strictly so over any interval containing zero).

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\(^2\) These properties are explicit or implicit in Kahneman and Tversky (1979). They are stated explicitly in Bowman, Minehart, and Rabin (1999).

\(^3\) The piecewise linear utility is used, for example, by Benartzi and Thaler (1995) and Barberis, Haung, and Santos (2001).
Köbberling and Wakker (2003) proposed the S-shaped utility function

\[
v(x) = \begin{cases} 
(1 - e^{-\alpha x})/\alpha & x \geq 0 \\
\lambda(e^{\beta x} - 1)/\beta & x < 0
\end{cases}
\]

with \(\alpha, \beta > 0\) and \(\lambda \geq 1\). \(10\)

Essentially the same function was introduced by Schmidt and Zank (2002) without the normalizing \(\alpha\) and \(\beta\) in the denominators. The functions are cardinally the same for \(\lambda_{SZ} = \lambda_{KW} \cdot \alpha_{KW} / \beta_{KW}\).

One advantage of this utility function is that marginal utility remains bounded while marginal TK utility becomes unbounded as \(x\) approaches 0 either from above or below.

While CPT does not assume risk aversion, it does assume that choices display loss aversion. Unlike risk aversion, there is no firm agreement on the precise meaning of loss aversion. At least four concepts that have appeared in the literature are described below. Köbberling and Wakker (2005) used \(\lambda\) as defined in property (d) as a measure of loss aversion. However, \(\lambda\) could be one and the function would still be loss averse by other definitions. When \(\lambda > 1\), there is a kink in the utility function so \(\lambda\) might be more aptly be related to first-order risk aversion. A more common definition of loss aversion is the rejection of all fair bets or all symmetric fair bets. This is an important property for equilibrium, but it really only serves to make the value function behave in a risk averse fashion as all concave utility functions reject all fair bets. In particular, Kahneman and Tversky originally defined loss aversion as the rejection of all symmetric fair bets. Strict KT loss aversion is the strict rejection of all risky symmetric fair bets.

**Theorem 12.2: Kahneman-Tversky Loss Aversion and Symmetric Fair Gambles.** All symmetric fair gambles are rejected in favor of the status quo by an S-shaped utility function with \(v(0) = 0\) if and only if the utility function displays Kahneman-Tversky Loss aversion, \(4\)

\[
v(-x) + v(x) \leq 0, \quad \forall x > 0.
\]

If the inequality in (11) is strict, then the status quo is strictly preferred.

**Proof:** A symmetric fair bet is defined as one in which for every outcome \(x\), there is a corresponding outcome, \(-x\), with the same probability. The expected utility of a symmetric fair bet is therefore

\[
\sum \pi_i v(x_i) = \sum_{x_i > 0} \pi_i [v(x_i) + v(-x_i)] \leq 0.
\]

If there is an outcome of 0, it contributes \(v(0) = 0\) to expected utility and can be ignored. The inequality holds because each term in the sum is nonpositive. If (11) holds strictly, then (12) also holds strictly provided \(\lambda\) has some risk so there is at least one pair of nonzero outcomes.

To prove the necessity of (11), note that a 50-50 bet receiving \(\pm x\) is a symmetric fair bet so \(\frac{1}{2}[v(-x) + v(x)] \leq 0\) if this bet is to be rejected. As this must be true for all \(x\), (11) follows immediately. If the bet is strictly rejected, then (11) holds strictly.

Note as an aside that the TK utility function, defined in (9), is not KT loss averse except for \(\alpha = \beta, \lambda \geq 1\). Condition (11) is satisfied only if \(x^{\alpha - \beta} < \lambda \beta / \alpha\) for all positive \(x\). However, this inequality must be violated either for sufficiently small \(x\) when \(\alpha < \beta\) or for sufficiently large \(x\) when \(\alpha > \beta\). For \(\alpha = \beta\), TK utility is KT loss averse for any \(\lambda \geq 1\). The Köbberling and Wakker utility function in (10) is KT loss averse if \(\beta \leq \lambda \alpha\).

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4 Because of the differing definitions for loss aversion, KT loss aversion is also called symmetric-bet aversion for obvious reasons. In subsequent work Kahneman and Tversky used the condition \(v(-x) + v(x) \leq v(-y) + v(y), \quad \forall 0 \leq y \leq x\), or the equivalent definition for differentiable utility \(v'(-x) \geq v'(x), \forall x > 0\). This is a stronger condition as setting \(y = 0\) ensures (11). This condition is also called increasing symmetric-bet aversion.
While KT loss aversion is sufficient to reject all fair symmetric gambles, it is insufficient to reject all fair gambles. This is also true of the stronger condition in footnote 4. A lottery whose payoffs are \{-25, -1, 9\} with probabilities of \{.25, .05, .70\} is fair with an expected payoff of 0. The TK utility with \(\alpha = \beta = 0.5\) and \(\lambda = 1.25\) satisfies both KT loss aversion and the stronger condition in footnote 4. However, it assigns it a utility 0.475 to the gamble, which is above the status quo of zero.

Subsequent work has developed stronger definitions of loss aversion. In particular, Neilson (2002) proposed\(^5\)

\[
\text{Weak Loss Aversion: } v(z)/z \geq v(y)/y \quad \forall z < 0 < y
\]

\[
\text{Strong Loss Aversion: } v'(z) \geq v'(y) \quad \forall z < 0 < y.
\]

KT loss aversion only compares the utilities of gains and losses of the same size. Both of Neilson’s definitions compare the utilities of gains and losses of different sizes. Weak loss aversion requires that the average utility per unit gain must be less that the average disutility per unit loss for a loss of the same size. The stronger definition requires the same of marginal utility.

Strong loss aversion implies weak loss aversion in general. However, for S-shaped utility they are equivalent definitions (obviously provided that utility is differentiable). Both conditions are stronger than KT loss aversion.

**Theorem 12.3: Loss Aversion Relations.** Strong loss aversion implies weak loss aversion for differentiable utility functions. They are equivalent for differentiable S-shaped utility functions. Either type implies KT loss aversion but not vice versa.

**Proof:** \(\text{Strong} \Rightarrow \text{Weak} \): Because \(v(0) = 0\), and \(v\) is differentiable

\[
v(y) - v(0) = \int_0^y v'(z)dz \leq v'_{\max} (y)\int_0^y dz = v_{\max}(y)y
\]

where \(v'_{\max} (y) \equiv \max_{0<z<y} v'(z)\)

\[
v(0) - v(x) = \int_x^0 v'(z)dz \geq v'_{\min} (z)\int_x^0 dz = -v'_{\min}(z)z
\]

where \(v'_{\min} (z) \equiv \min_{z<x<0} v'(x)\).

From (14), \(v'_{\min} (z) \geq v'_{\max} (y), \forall z < 0 < y\), so

\[
\frac{v(z)}{z} \geq v'_{\min} (z) \geq v'_{\max} (y) \geq \frac{v(y)}{y}
\]

verifying (13).

\(\text{Weak} \Rightarrow \text{Strong}\) for S-shaped \(v\): Suppose (13) holds but (14) does not. Then there must be a pair of values, \(z_0 < 0 < y_0\) with \(v'(y_0) > v'(z_0)\). For S-shaped utility, \(v\) is concave for positive arguments, and \(v(0) = 0\); therefore, \(v_0'v'(y_0) \leq v(y_0)\). For negative arguments using L’Hospital’s rule

\[
\lim_{z \to -\infty} \frac{v(z)}{z} = \lim_{z \to -\infty} \frac{v'(z)}{1} \leq v'(z_0).
\]

The inequality follows from the convexity of \(v\) for negative arguments assuring that \(v'\) is non-decreasing. Therefore, there must be some \(z_1 < z_0\) such that \(v(z_1)/z_1 \leq v'(z_0)\). Combining, we have

\[
\frac{v(z_1)}{z_1} \leq v'(z_0) < v'(y_0) \leq \frac{v(y_0)}{y_0}.
\]

\(^5\) Bowman, Minehart, and Rabin (1999) actually assumed a stronger version with \(v'(z) \geq 2v'(y) \quad \forall z < 0 < y\).
This contradicts the assumed Neilson loss aversion.

Weak $\Rightarrow KT$: If (13) holds, then for $x > 0$, $v(−x)/(−x) ≥ v(x)/x$ so $−v(−x) ≥ v(x)$ which is (11). Obviously strong loss aversion which implies weak loss aversion also implies TK loss aversion.

$KT \not\Rightarrow Weak$: For TK utility with $\alpha = \beta$, (11) is satisfied, and $v(y)/y = y^{\alpha−1}$. This expression is unbounded for small $y$ unless $\alpha = 1$, so it clearly exceeds $v(z)/z$ for some negative $z$ values. So Neilson loss aversion does not hold. And if weak loss aversion does not hold strong loss aversion cannot either.

As just shown, the TK utility function does not display weak loss aversion except when $\alpha = \beta = 1$. The Köbberling and Wakker utility in (10) is weakly loss averse if $\alpha ≥ \beta$ ($\lambda ≥ 1$). If $v$ is not S-shaped, then it can satisfy weak loss aversion (13) without satisfying strong loss aversion (14). A simple example is

$$v(x) = \begin{cases} 2x & x ≤ 0 \\ x & 0 ≤ x ≤ 1 \\ 3x−2 & 1 ≤ x ≤ 2 \\ x+2 & 2 ≤ x. \end{cases} \quad (19)$$

This function is piecewise linear and continuous but not S-shaped. For positive $x$, it is first convex then concave. It displays weak but not strong loss aversion because marginal utility is 2 for negative outcomes but is 3 at positive outcomes between 1 and 2.

**Theorem 12.4: Weak Aversion and Fair Gambles.** In the absence of probability weighting, weak loss aversion is both necessary and sufficient for the rejection of all fair gambles in favor of the status quo.

**Proof:** Sufficiency. From (13) there exists a positive number $c$ such that for all $z < 0 < y$

$$\frac{v(y)}{y} ≤ c ≤ \frac{v(z)}{z} \Rightarrow \begin{cases} 0 < v(y) ≤ cy \quad \forall y > 0 \\ v(z) ≤ cz < 0 \quad \forall z < 0. \end{cases} \quad (20)$$

Therefore, for any gamble $\mathbb{E}[v(\tilde{x})] ≤ \mathbb{E}[c\tilde{x}]$, and so any gamble with an expected payoff of 0 has non-positive expected utility. If the loss aversion is strict, then the expected utility must be negative if $\tilde{x}$ has any variation.

Necessity. Assume (13) is not necessary. That is assume that the status quo is preferred for all fair gambles, but there are two values, $z_0 < 0 < y_0$ with

$$\frac{v(z_0)}{z_0} < \frac{v(y_0)}{y_0}. \quad (21)$$

Consider the gamble that pays $z_0$ with probability $\pi \equiv y_0/(y_0 − z_0)$ and $y_0$ with probability $1 − \pi$. It is a fair gamble, but its expected utility is

$$\mathbb{E}[v(\tilde{x})] = \pi v(z_0) + (1 − \pi)v(y_0) = \frac{v_0v(z_0) − z_0v(y_0)}{y_0 − z_0} = \frac{v_0v(z_0) − v(y_0)}{y_0 − z_0} \left[ \frac{v(z_0)}{z_0} − \frac{v(y_0)}{y_0} \right]. \quad (22)$$

The term in brackets and $z_0$ are negative by assumption while the other factors are positive so the expected utility is positive. This contradicts the assumption that the status quo was preferred.

The verbal descriptions of loss aversion are given in terms of expressed preferences — the rejection of gambles. However, the inequality property in (11), (13), and (14) depend only on the utility function ignoring probability weighting. So the verbal descriptions of loss aversion must be interpreted as applying when probability weighting is not used or when “fairness” is
defined in terms of the probability weights rather than the actual probabilities.

When probability weighting is employed, the results above do not necessarily hold. Under probability weighting, a fair binomial \( \pm x \) bet is accepted if \( \Omega(0.5)v(-x) + [1 - \Omega(0.5)]v(x) > 0 \). So KT loss aversion does not guarantee the rejection of all 50-50 fair bets. KT loss aversion together with \( \Omega(0.5) \geq 0.5 \) are sufficient conditions for the rejection of all fair \( \pm x \) gambles. When \( \Omega(0.5) < 0.5 \), additional assumptions are needed. For example, consider any utility function with \( v(-x) = -\lambda v(x), \forall x > 0 \). Such utilities are KT loss averse, but they reject fair 50-50 gambles only if \( \Omega(0.5) \geq 1/(1+\lambda) \).

Similarly, weak loss aversion is not sufficient by itself for rejection of all fair bets under probability weighting. Consider a gamble that pays \( a > 0 \) with probability \( k/(1+k) \) and \(-ka\) with probability \( 1/(1+k) \). This gamble is fair, but it is rejected only if

\[
\Omega(((1+k)^{-1})v(-ka) + [1-\Omega((1+k)^{-1})]v(a) \leq 0 \quad \Rightarrow \quad \Omega((1+k)^{-1}) \geq \frac{v(a)}{v(a) - v(-ka)}.
\]  

The restriction in (13) can be restated as \( v(-ka) \geq -kv(a) \) for \( a, k > 0 \). When this restriction is just binding, \( v(-ka) = -kv(a) \), rejection requires that

\[
\Omega((1+k)^{-1}) \geq (1+k)^{-1}.
\]  

As \( k \) is an arbitrary positive number, (24) can only be satisfied if \( \Omega(P) \geq P, \forall P \); that is, when the probability weighting function is pessimistic.

**The Complete Market CPT Portfolio Problem**

The portfolio problem for CPT is quite similar to that for RDU. Portfolio returns are provisionally ordered as in (2) so the probability weights can be determined. The reference point, denoted as \( \hat{w} \), separates the concave and convex portions of the utility function and marks its kink, if any. When the portfolio has a value of \( \hat{w} \) in state \( s \), then \( x_s = 0 \). In most portfolio models, the reference point is current wealth, current wealth increased at the risk-free rate, or some function of future wealth like its expectation.

The Lagrangian for this problem is

\[
L = \sum_{s=1}^{S} \omega_s v(x_s) + \eta \left[ w_0 - \sum_{s=1}^{S} q_s (\hat{w} + x_s) \right] + \sum_{s=1}^{S-1} \kappa_s (x_{s+1} - x_s).
\]  

To simplify the notation, define \( B \equiv w_0 - \hat{w} \sum q_s = w_0 - \hat{w}/(1+r_f) \), then the budget constraint can be expressed as \( \sum q_s x_s = B \).

The first-order conditions are as given in the previous chapter with one modification. If the utility function does have a kink at \( x = 0 \), then in addition we require that for any state in which \( x_s = 0 \),

\[
\left. v'(x_s^+) \right|_{x_s=0} \equiv \lim_{x \downarrow 0} v'(x) \leq \frac{\eta q_s}{\omega_s} \leq \left. v'(x_s^-) \right|_{x_s=0} \equiv \lim_{x \uparrow 0} v'(x).
\]  

That is, there is not a unique value of the \( \omega \)-SDF for which the optimal portfolio outcome is \( x_s = 0 \). Rather there is a range of \( \omega \)-SDF values for which the optimal portfolio has neither a gain nor loss.

Before solving the portfolio optimization problem, we must deal with one other feature of S-shaped utility; the objective function is not concave. This does not mean the problem does not have a solution; however, when the optimal portfolio seeks unbounded positions, there is no equilibrium, and the standard pricing relations will not hold. Although loss aversion assures that all fair bets are
rejected, it is insufficient to guarantee bounded optimal portfolios when some opportunities have positive excess expected rates of return.

As an example, consider an investor with the loss-averse S-shaped utility function

\[ v(x) = \begin{cases} 
  x & x \geq 0 \\
  3x & -2 \leq x \leq 0 \\
  2x - 2 & x \leq -2 
\end{cases} \]  

(27)

with a reference point at current wealth. The economy has two states with equal probability weights and state prices of \( q_a = \frac{1}{4}, q_b = \frac{3}{4} \). The investor purchases \( n_s \) Arrow Debreu securities for state \( s \) giving \( x_s = n_s - w_0 \). The two portfolio demands are related by the budget constraint, \( n_s = \frac{1}{3} (w_0 - \frac{1}{4} n_s) \). For any portfolio with \( x_a > 6 \), the outcome in state \( b \) is on the shallower portion of the utility for losses; \( x_b < -2 \). The expected utility of all such portfolios is

\[ 0.25 v(a) + 0.75 v(b) = 0.25 \left( n_s - w_0 \right) + 0.75 \left( w_0 - \frac{1}{4} n_s \right) 3 \left( w_0 - n_s / 6 \right) - 2 = \frac{1}{6} (n_s - w_0) - 2. \]  

(28)

This is strictly increasing in \( n_s \) so the optimal portfolio is unbounded.

There are a number of ways to eliminate this problem. Long before CPT was developed, Friedman and Savage proposed that utility functions had a convex portion, but it was in the midrange of outcomes and surrounded by two concave portions. For a utility function like that, unbounded gambles will not be optimal. If a strictly concave portion is added to the lowest part of the CPT utility function, the optimal portfolio will not be unbounded.

Another fix is to assume that actions are constrained. For example, long and short position limits obviously prevent a portfolio from having unbounded holdings. Similarly, a constraint on a maximum allowable loss, such as not allowing portfolios that can result in negative wealth, would bound portfolio positions. A maximum allowable loss can be considered a special case of Friedman Savage utility where \( v(\Delta w) = -\infty \) for \( \Delta w < -w_0 \).

A third alternative is to restrict the utility function. A sufficient condition to ensure a bounded maximum for a portfolio is Extreme-Risk Avoidance.

**Definition: Extreme-Risk Avoidance (XRA).** An increasing utility function with \( v(0) = 0 \) displays extreme-risk avoidance if

\[ \limsup_{x \to \infty} \frac{v(x)}{v(\kappa - kx)} = 0 \quad \forall k > 0, \forall \kappa. \]  

(29)

Note that XRA is the avoidance of extreme risks rather than the avoidance of all risks in the extreme.

As \( v(x) \) is strictly increasing, \( v(x) \) must be positive as \( x \to \infty \). So a utility function can display XRA only if \( v(x) \to -\infty \) as \( x \to -\infty \). The Köbberling and Wakker utility function in (10) does not display XRA for any parameters because \( \lim_{x \to -\infty} v(x) = -\lambda / \beta \). The TK utility function displays XRA if and only if \( \alpha < \beta \) because \( v(x) / v(\kappa - kx) \sim -k^{-\alpha} x^{\alpha - \beta} / \lambda \).

Because portfolios have linear tradeoffs among their outcomes across states, extreme portfolios can have very large gains in some states only if they have proportionally large losses in other states. As shown in the next theorem, XRA ensures that the disutility of the large losses more than offsets the utility of the gains in any extreme portfolio.

**Theorem 12.5: Bounded Optimal Portfolios with Extreme Risk Avoidance.** If an investor has extreme risk avoidance, then the optimal portfolio has bounded positions in every asset.

---

6 Because \( v \) is an increasing function, \( \lim_{x \to \infty} v(x) \) and \( \lim_{x \to -\infty} v(x) \) both exist in the extended reals, \([-\infty, \infty]\). However, the limit of the ratio need not exist. The supremum limit always exists.
Proof: The budget constraint puts an upper bound on the worst return in terms of the best return. Order the returns with \( x_1 \leq x_2 \leq \cdots \leq x_S \). Then from the budget constraint \( B = \sum_{s=1}^{S} q_s x_s \geq x_1 \sum_{s=1}^{S-1} q_s + q_S x_S \), so \( x_s \leq \Lambda \cdot (B - q_s x_s) \) where \( \Lambda^{-1} = \sum_{s=1}^{S} q_s \). Because utility is increasing, and the portfolio outcomes are weakly ordered, the decision-weighted utility of this portfolio is

\[
E_{\Omega} [v(x)] \equiv \sum_{s=1}^{S} \omega_s v(x_s) \leq \omega_1 v(x_1) + v(x_S) \sum_{s=2}^{S} \omega_s \leq \omega_1 v(\Lambda \cdot (B - q_S x_S)) + v(x_S) \sum_{s=2}^{S} \omega_s .
\]  

(30)

When the highest outcome, \( x_S \), is increased, the second term increases and the first term becomes more negative. \( \Lambda \), \( B \), and the weights, \( \omega_s \), are fixed, so if the utility function has XRA, \( v(\Lambda B - \Lambda q_S x_S) \gg v(x_S) \), and the right-hand side of (30) will be unboundedly negative, and the portfolio cannot be optimal.

Theorem 12.5 shows that XRA limits optimal portfolios to hold only bounded positions. This means that aggregated optimal positions will also be bounded so that an equilibrium is possible. The properties of the equilibrium and whether or not a representative investor exists still need to be determined. As always the latter would indicate that the market portfolio is an efficient allocation and can therefore be used as the basis of a SDF. The next two theorems address this issue.

Theorem 12.6: Portfolio Ordering. For any two states that are equally probable, the optimal portfolio of an investor with increasing utility who uses decision weights has at least as high a return in the state with the smaller state price.

Proof: Consider two states, \( a \) and \( b \), with equal probabilities \( \pi_a = \pi_b \). With no loss of generality take \( q_a > q_b \). Now assume that the proposition is false and \( x^*_a = h > \ell = x^*_b \). The otherwise identical portfolio with \( x_a = \ell \) and \( x_b = h \) is affordable because \( q_a > q_b \). Swapping these two returns changes the order of the outcomes across states; however, as the two states have the same probabilities and the weighting function depends only on the cumulative probabilities, only the decision weights for those two states will be affected and they will simply be swapped. Therefore, the decision-weighted expected utility for the altered portfolio will be equal to that for the originally assumed optimal portfolio. The cost of the altered portfolio is \( (q_a - q_b)(h - \ell) \) less than the cost of the original portfolio. This extra can be invested in the risk-free asset increasing the return realized in every state. Since this will not alter the ordering of the outcomes, the decision weights remain the same and this final portfolio will have a higher decision-weighted expected utility than the originally assumed optimal portfolio. Therefore, the original portfolio cannot have been optimal, and we must have \( x^*_a \leq x^*_b \).

This theorem requires only that utility be increasing and not necessarily concave; therefore, it applies both to RDU with concave utility and CPT with S-shaped utility. This theorem reduces the work needed to determine an optimal portfolio because some orderings need not be considered. But more importantly, it is the basis for a much more useful theorem.

Theorem 12.7: Weak Monotonicity of Optimal Decision-Weighted Portfolios. Assume a complete market with equally likely states. The returns on the optimal portfolio of any investor with increasing utility is decreasing in the objective-probability SDF or price-probability ratio, \( m_s = q_s / \pi_s \). The returns are strictly decreasing over ranges where \( q_s / \omega_s \) is strictly decreasing and constant over ranges where \( q_s / \omega_s \) is increasing or constant.

Proof: Weak monotonicity of the returns in \( q/\pi \) follows directly from Theorem 12.6 because all states are equally probable. We need only ascertain when the ordering is strict. Consider a range where \( q_s / \omega_s \) is increasing or constant and, contrary to the proposition, that \( x^*_s < x^*_{s+1} \). From the first order conditions in (26), the multiplier \( \kappa_s \) must be zero when the portfolio returns differ so
\begin{align}
&x_s^* = v^{-1} \left( (\eta q_s - \kappa_{s-1})/\omega_s \right) \geq v^{-1} \left( \eta q_s / \omega_s \right) \\
&x_{s+1}^* = v^{-1} \left( (\eta q_{s+1} + \kappa_{s+1})/\omega_s \right) \leq u^{-1} \left( \eta q_{s+1} / \omega_s \right).
\end{align}

The inequalities follow because the two multipliers, \( \kappa_{s-1} \) and \( \kappa_{s+1} \), are nonnegative and \( v^{-1} \) is a decreasing function. But the monotonicity of \( u^{-1} \) also implies that
\begin{equation}
\begin{aligned}
x_{s+1}^* &\leq v^{-1} \left( \eta q_{s+1} / \omega_s \right) \leq v^{-1} \left( \eta q_s / \omega_s \right) \leq x_s^*
\end{aligned}
\end{equation}
which is a contradiction so \( x_s^* = x_{s+1}^* \) when \( q_s/\omega_s \leq q_{s+1}/\omega_{s+1} \).

Now consider a range where \( q_s/\omega_s \) is strictly decreasing and, contrary to the proposition, that \( x_s^* = x_{s+1}^* = x \). Suppose the portfolio is altered to receive \( \varepsilon \) less in state \( s \) and \( q_s \varepsilon /q_{s+1} \) more in state \( s+1 \). This altered portfolio has the same cost as the original so it is affordable. The change in utility is
\begin{equation}
\begin{aligned}
\Delta_R \left[ v(x) \right] &= \omega_s [v(x-\varepsilon) - v(x)] + \omega_{s+1} [v(x + q_s \varepsilon / q_{s+1}) - v(x)] \\
&\approx v'(x) q_s \varepsilon \left[ \omega_{s+1} / q_{s+1} - \omega_s / q_s \right] \geq 0
\end{aligned}
\end{equation}
which is positive. The ratio \( \omega_{s+1}/q_{s+1} \) is the reciprocal of \( q_s/\omega_s \) so it is strictly increasing. Again this is a contradiction so we must have \( x_s^* < x_{s+1}^* \) when \( q/\omega \) is decreasing.

The monotonicity of RDU and CPT portfolios in the objective SDF need not obtain when states have different probabilities. However, this result can be extended to such markets provided investors can create financial contracts that are fair-value sub-state bets. When such financial contracts can be created, any state with probability, \( \pi \), and state price, \( q \), can be partitioned into two or more sub-states with proportional probabilities and sub-state prices; i.e., \( q'/\pi' = q/\pi \) for all sub-states of the original state. The theorem can be applied to these equally probable sub-states and then extended back to the original states by aggregation.\(^7\)

Another case of no little interest is a state space with a continuous probability distribution containing no atoms. For such a distribution, the state index is somewhat arbitrary and can always be rescaled so that the density function is constant.\(^8\) Therefore, for complete market models with a continuous state space, the optimal portfolio for any risk-averse RDU investor must have returns that are weakly decreasing in the objective price-probability ratio \( q/\pi \) with constant or strictly decreasing returns depending on whether \( q/\omega \) is decreasing or weakly increasing.

Figure 12.1 illustrates the three ratios for a Tversky-Kahneman weighting function. The objective-probability SDF, \( q/\pi \), is falling (by construction). The ratio of the probability weight to the natural probability, \( \omega/\pi \), is U-shaped because the decision weights are larger than the actual probabilities for the outcomes in both tails. The decision-weight SDF, \( q/\omega \), is the quotient of the two and has an inverted U shape. It must be decreasing in the range where \( \omega/\pi \) is rising, but is
increasing when $\omega/\pi$ is sufficiently steeply declining. The typical case is illustrated with $q/\omega$ increasing for the smallest values of the market’s return. In such an economy the optimal portfolio of a risk-averse RDU maximizer has the same return in all of the poorest outcome states. In the better states, returns are increasing with the state just as for an EUT maximizer.

An important implication of Theorem 12.7 is that with homogeneous objective beliefs, the market portfolio itself is objectively efficient in a complete market just as it is under EUT.

**Theorem 12.8: Objective Efficiency of Market Portfolio under CPT or RDU.**

Assume a complete market with equally likely states, all investors are risk-averse or have S-utility and have homogeneous objective beliefs. In addition, there is at least one risk-averse investor who uses objective probabilities. Then, in equilibrium, the market portfolio’s returns will be strictly decreasing in the objective SDF (price-probability ratio) and the market portfolio will be objectively risk-averse efficient.

**Proof:** From Theorem 12.7, the returns on each investor’s optimal portfolio are weakly ordered inversely to the objective price-probability ratio. Because the market portfolio is a convex combination of these optimal portfolios, its returns must also be weakly decreasing in the ratio. Now assume this monotonicity is not strict; that is, assume there are two states with different price-probability ratios but equal market returns. The risk-averse investor using objective probabilities does hold a strictly monotone portfolio with a higher return in the better state; therefore, if the market is to clear with an equal return in the two states, some other investor must hold a portfolio with a smaller return in the better state. But this contradicts Theorem 12.7. So in equilibrium, the market portfolio’s returns must be strictly decreasing in the objective price-probability ratio and therefore optimal for some strictly risk-averse utility function.

Theorem 12.8 shows that under CPT or RDU the only additional requirement for there to be a representative investor who holds the market portfolio is that all states are equally likely. The other assumptions in the theorem, like homogeneous beliefs and the absence of market frictions, are standard. This is significant because it assures the existence of a risk-averse representative agent who uses the objective probabilities. This means that price data alone cannot logically reject a risk-averse objective-probability, i.e. classical, equilibrium in favor of a CPT or RDU equilibrium arising from S-utility and/or probability-weighting. However, this statement has several caveats. First, the proposition does not prove that there is a unique representative agent so there may also exist one (or more) S-utility representative agents who do use probability

**Figure 12.1 Three State-Price Densities**

This figure illustrates the three state price densities: the state price per unit probability, the decision weight per unit probability, and the state price per unit decision weight.
weighting, and this latter representation might be viewed as statistically more likely. Second, other information such as asset holdings, trades, volume, etc. might be inconsistent with a classical equilibrium. Third, this is a single-period result so there might be evidence in price dynamics that are inconsistent with a classical equilibrium. Finally, all of the analysis assumes homogeneous objective beliefs.

Theorem 12.8 remains valid even in an incomplete market provided investors are able to create financial contracts. The same reasoning used in Chapter 4 can be used to show all that is required is effectively complete markets — one in which all investor have the same shadow state prices for all states. Investors have incentives to introduce financial contracts until this is achieved. If market completion with financial assets is not allowed, then the inverse ordering between optimal portfolios and the ratio, \( q/\pi \), need not hold. Of course, that property need not hold amongst EUT investors using the true probabilities either.

One important property of a complete market under EUT does not carry over for CPT investors. Namely, a complete market equilibrium does not necessarily result in a Pareto-optimal risk sharing. This is true even if investors have identical beliefs and utility functions. For example, consider the following two-state economy. The state prices and probabilities are \( q = (0.8, 0.2)' \) and \( \pi = (0.75, 0.25)' \). Consider investors with TK-utility parameters \( \alpha = 0.4, \beta = 0.6, \lambda = 2 \), and a reference level equal to their initial wealth of 1 who do not use probability weighting. Optimal time-1 consumption is \((65/64, 15/16)'\) with gains/losses relative to the reference level of \((1/64, -1/16)'\). This satisfies the first order conditions

\[
\frac{\pi_1 v'(x_1)}{q_1} = 0.75 \cdot 0.4 \cdot \left(\frac{1}{64}\right)^{0.4-1} = 0.8 = \frac{0.25 \cdot 2 \cdot 0.6 \cdot \left(\frac{1}{16}\right)^{0.6-1}}{0.2} = \frac{\pi_2 v'(x_2)}{q_2}.
\]

It also satisfies the second order condition, which is important to check, with a non-convex problem like this. This choice achieves a utility of

\[
\pi_1 v(x_1) + \pi_2 v(x_2) = 0.75 \cdot \left(\frac{1}{64}\right)^{0.4} + 0.25 \cdot 2 \cdot \left(\frac{1}{16}\right)^{0.6} = 0.047.
\]

Two identical investors can increase their utility by making a side bet flipping a fair coin when the second state occurs. This happens because consumption is a loss relative to the reference level and in the risk seeking portion of their utility. If they bet \( b \) on the outcome, their consumption in state 2 is \( 15/16 \pm b \) with a gain and loss of \(-1/16 \pm b\). A bet of 0.1127 yields consumption of either 1.0501 or 0.8248. The gain and loss are 0.0502 and \(-0.1752\). The probability of either of these outcomes is 0.125, half the probability of state 2. The sub-state prices are 0.1, half the state price of state 2.\(^9\) With the bet, their expected utility is

\[
\pi_1 v(x_1) + \pi_2 v(x_2) + \pi_2 v(x'_{2a}) + \pi_2 v(x'_{2b}) = 0.75 \cdot \left(\frac{1}{64}\right)^{0.4} + 0.125 \cdot (0.050)^{0.4} - 0.125 \cdot 2 \cdot (0.175)^{0.4} = 0.920. \quad (36)
\]

There is an interior optimum to this problem because the utility function used has extreme risk avoidance as defined earlier, \( v(x)/-v(-x) = \frac{1}{2} x^{-0.2} \rightarrow 0 \) as \( x \rightarrow \infty \).

This process of introducing side bets when utility has convex regions serves to convexify the utility function. This is also illustrated in the figure. The utility function is shown in black. With no bet, consumption in both states \( 2a \) and \( 2b \) is \( 15/16 \) with a “loss” of \(-1/16\) relative to the reference point of 1. This gives a contribution to utility of \(-0.38\). The blue line illustrates a bet of \( \pm 0.07 \) resulting in consumption of \(-13.25 \) or 0.0075. These are the \( x \) coordinates at the ends of the line. The average utility contribution is \(-0.23\), the \( y \) coordinate at the midpoint of the line.

\(^9\) The added gamble of the coin flip is completely independent of the economy and should not change prices if the two CPT investors are small relative to the market. So the two sub-state prices must sum to 0.2, and by symmetry they should be equal.
The optimal bet of 0.1127 is illustrated by the red line. The two utility realizations are \(-0.703\) and \(0.302\) with an average of \(-0.201\). Increasing the bet further to 0.225 decreases the average contribution to utility as shown by the green line.

But the derivative asset creation doesn't need to end there. The substate with the loss of \(-0.703\) can also be divided. A bet of 0.268 gives a gain and loss of 0.024 and \(-0.077\) for an average contribution of utility of \(-0.053\) in place of the already improved \(-0.201\). This process could, of course, continue.

**CPT and the CAPM**

As explored in Chapter 5, mean-variance analysis and the CAPM are valid when utility is quadratic or when asset returns are drawn from the class of elliptical distributions, including the normal. A quadratic function is possible in RDU, but obviously cannot have the desired S-shape for CPT. However, most of the mean-variance results of elliptical distributions, including the CAPM, continue to be valid with probability weighting with or without S-shaped utility. This can be summarized as follows.

**Theorem 12.9: Objective CAPM under RDU (including CPT).** Assume (i) the returns on all assets are elliptically distributed, (ii) there are no transactions costs or differential taxes, (iii) borrowing and lending (or if there is no risk-free asset, shorts sales) are unrestricted, and shares are infinitely divisible, (iv) all investors have strictly increasing utility, homogeneous objective beliefs, and a common single-period horizon, and (v) they evaluate outcomes using a strictly increasing and once differentiable probability-weighting function. Then, provided an equilibrium exists, two-fund separation obtains with the CAPM relation between the objective means and covariances.

**Proof:** As shown in Chapter 5, elliptical variables are completely characterized by their objective mean and variance. The return on any portfolio can be expressed as a translated and
scaled variable $\tilde{r}_w = \mu_w + \sigma_w \tilde{\rho}$ where $\tilde{\rho}$ is a standardized elliptical variable with zero mean and unit variance.

Define the derived mean-variance evaluation function for a particular weighting function, $\Omega$, as

$$V_{\Omega}(\mu, \sigma) \equiv E_{\Omega}[u(\tilde{r}_w(\mu, \sigma))] \equiv \int_{-\infty}^{\infty} u(R(\rho))d\Omega(F(\rho))$$

(37)

where $R(\rho) \equiv \mu + \sigma \rho$, and $F$ is the cumulative distribution for $\rho$. $^\text{10}$ $R(\rho)$ is strictly increasing in $\mu$, $u$ is strictly increasing, and $d\Omega(F(\rho))$ is nonnegative; therefore, $V_{\Omega}$ must be strictly increasing in $\mu$. Therefore, regardless of individual weighting functions, all portfolios are characterized by their objective means and variances, and each investor’s optimal portfolio is on the objective mean-variance efficient frontier. As individual demands all lie on the mean-variance efficient frontier, the aggregate demand must as well so in equilibrium the market portfolio is mean-variance efficient based on its objective (i.e., unweighted) distribution. The CAPM follows immediately.

The only differences between this theorem and the standard case are probability weighting as given in assumption (v) and the possible absence of risk aversion. Probability weighting does not affect the result at all. Mean-variance analysis and the CAPM remain correct for any valid weighting function. We do need to assume that demands are finite so that an equilibrium exists, but this will be true under extreme-risk aversion as shown previously. $^\text{11}$

Of course, the CAPM of this theorem need not be the same as the one that would prevail if all investors used objective probabilities and were risk averse. In particular, typical probability weighting functions increase the market price of risk as they emphasize the extreme outcomes making investors more reluctant to take on the risk of the market if they are risk averse or Neilson loss averse. Furthermore, the makeup of the market portfolio itself will typically change as a different market price of risk will alter the point of tangency of the borrowing-lending line.

Barberis and Huang (2008) proved a similar theorem for the special case with a multivariate normal distribution and investors with identical TK utility (with $\alpha = \beta$), identical TK weighting functions, and a zero-utility reference return equal to the risk-free rate. On the other hand, De Giorgi, Hens, and Levy (2004) show that an equilibrium does not exist if investors have heterogeneous TK utility functions each with $\alpha_k = \beta_k$. $^\text{12}$ There is no equilibrium precisely because such investors lack extreme-risk avoidance and desire unbounded positions in the tangency portfolio. An equilibrium does exist with no probability weighting and the S-shaped piecewise exponential utility function given in (10). Together, these assumptions are operationally equivalent to extreme-risk avoidance and ensure that high leverage is not desirable.

$^\text{10}$ The utility function used here defined on rates of return is $u(r_w) = v(w_0 (1 + r_w) - \hat{w})$, where $v$ is the CPT utility function defined on payoffs with a reference level of $\hat{w}$ for the zero-utility level.

$^\text{11}$ In fact, this is also true in the standard CAPM. The assumption of risk aversion there merely proves that investors prefer to hold the minimum-variance portfolio among all portfolios with the same expected payoff. This does not guarantee that their optimal portfolio has a finite mean and variance. If an investor’s mean-standard-deviation indifference curves asymptote to a line with a slope less than the Sharpe ratio, they will have infinite demand even in the standard CAPM.

$^\text{12}$ Some heterogeneity is typically required for the absence of any equilibrium. If all investors are identical, then there usually is a trivial equilibrium where each investor holds exactly the market portfolio because that is the only symmetric feasible strategy. This is the fundamental distinction between the Barberis and Huang (2008) and the De Giorgi, Hens, and Levy (2004) models. This symmetry is also the basis for all representative agent models.
Loss Aversion and the Endowment Effect

Four definitions of loss aversion that have appeared in the literature were discussed earlier. KT loss aversion and weak and strong Neilson loss aversion all concern the rejection of fair gambles so they are related to standard risk aversion. Köbberling and Wakker’s measure of risk aversion depends only on the kink in utility at the reference level and is similar to first-order risk aversion. But loss aversion goes beyond these ideas.

Loss aversion or a very similar effect is even evident in risk-free contexts. Thaler (1980) termed the effect in the absence of risk the endowment effect by. It has been tested by many researchers. The basic test involves giving half the subjects some small gift and determining the price they at which they are willing to sell it back, the acceptance price, or in finance terms, the asking price. The other half receives no gift, and the price at which they are willing to purchase it is determined. This is the willingness to pay or the bid price. The first group’s average asking price is always larger than the second group’s average bid price.

The classic experiment is due to Kahneman, Knetsch, and Thaler (1990). One group was given coffee mugs that sold for $6 in the Cornell bookstore. For this group, the median asking price demanded was $5.25 in each of four trials. A similar group who were not given mugs had a median bid price of $2.25 (3 times) or 2.75 (once). The two groups then switched roles with $3.98 pen sets. The median bid and ask prices were $0.75 and 1.75-2.50. The ratio of median ask to median bid was greater than 2 in all eight trials. In other experiments cited in their paper, the ratio of ask means to bid means varied from 1.4 to 16.5.

The lowest asking price for the mugs was $2.25. The highest bid price was $4.75. The supply and demand curves were upward and downward sloping, respectively, in each trial as expected. The market clearing price ranged from $4.25 to $4.75 in the four trials. In the latter case only one mug was traded; in one case with the lower market price, four mugs traded. In the pen experiment, the market clearing price was $1.25 in each trial and either four or five pens traded.

The Coase theorem says that, ignoring wealth effects, initial property rights are irrelevant to the final equilibrium allocation. Assuming the reservation values have similar distributions among those who had and didn’t have the prizes, the Coase theorem predicts that half the objects should be traded. In the two experiments, only 20% of the mugs and 41% of the pens actually traded. So the endowment effect seems to prevent what is generally considered the Pareto benefits of trade under standard conditions.

In a related experiment by Knetsch (1989), three groups were a) offered either a coffee mug or a chocolate bar, b) given a mug and later offered the opportunity to trade it for a chocolate bar, and c) given a chocolate bar and later offered the opportunity to trade it for a mug. In group a, 56% preferred the mug; in group b, 89% preferred the mug; and in group c, only 10% preferred the mug. This experiment indicates that the endowment effect is present even in the absence of money. Whether the endowment effect is the same as loss aversion or something different is a matter of debate.

In one experiment, Gächter, Johnson and Herrmann (2010) determined individual ask-bid ratios in trials like that just described. For 5% of the individuals the ratio was less than 1; it was equal to 1 for 7%. So 12% of the participants showed no or a negative endowment effect. The

---

13 The bid and ask prices were determined by assessing the willingness to purchase or sell in 25¢ increments.
14 Wealth effects should be negligible in these experiments. However, Hanemann (1991) has shown theoretically that, even without loss aversion or the endowment effect, the ask to bid ratio can exceed one by any amount if there is no close substitute for the good in question. As the goods in this experiment were readily available at the bookstore, this explanation would apply only if the participants were not considering outside alternatives but were narrow framing their experience within the experiment.
ratio was greater than 1 for 88%. It exceeded 1.2 for 81% of the sample and exceeded 1.5 for 67%. The median value was 2. This test provides individual estimates of \( \lambda \) completely devoid of risk and therefore of risk aversion provided utility is not too far from risk-neutral.\(^{15}\)

The same participants’ loss aversion was also tested on risky propositions. Each was offered six 50-50 lotteries. In each lottery, €6 was won for heads. The losses for tails ranged from €2 to €7 in steps of €1. A participant with probability weighting and KT utility would accept the gamble with a loss of €\( L \) if

\[
\Omega^{+}(\frac{1}{2}) \cdot 6^\alpha > 0 \quad \text{or} \quad \lambda < \Omega^{+}(\frac{1}{2})6^\alpha / \Omega^{-}(\frac{1}{2})L^\beta.
\]

The choices of each participant were recorded and their loss aversion parameter was estimated (36) based on the largest loss they would accept.\(^{16}\) Two utility functions with \( \alpha = \beta = 1 \) or \( \alpha = 0.95, \beta = 0.92 \) were used. Two probability weighting structures with \( \Omega^{+}(\frac{1}{2})/\Omega^{-}(\frac{1}{2}) = 1 \) or 0.86 were also used. The table below shows the estimated upper bound on the loss aversion parameter for the four cases.\(^{17}\)

### Table 12.1 Upper Bound Estimates for Loss Aversion Parameter

<table>
<thead>
<tr>
<th>Parameters</th>
<th>( \Omega^{+}(\frac{1}{2})/\Omega^{-}(\frac{1}{2}) = )</th>
<th>1</th>
<th>1</th>
<th>0.86</th>
<th>0.86</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha = )</td>
<td>1</td>
<td>0.95</td>
<td>1</td>
<td>0.95</td>
<td></td>
</tr>
<tr>
<td>( \beta = )</td>
<td>1</td>
<td>0.92</td>
<td>1</td>
<td>0.92</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>largest loss accepted</th>
<th>% of participants</th>
<th>maximum possible implied ( \lambda )</th>
</tr>
</thead>
<tbody>
<tr>
<td>none</td>
<td>1.84%</td>
<td>1.84%</td>
</tr>
<tr>
<td>€2</td>
<td>9.51%</td>
<td>2.580</td>
</tr>
<tr>
<td>€3</td>
<td>15.95%</td>
<td>1.997</td>
</tr>
<tr>
<td>€4</td>
<td>25.77%</td>
<td>1.532</td>
</tr>
<tr>
<td>€5</td>
<td>17.79%</td>
<td>1.248</td>
</tr>
<tr>
<td>€6</td>
<td>12.58%</td>
<td>1.055</td>
</tr>
<tr>
<td>€7</td>
<td>16.56%</td>
<td>0.857</td>
</tr>
</tbody>
</table>

The median value for \( \lambda \) is shown in the €4 row. This is less than the median value of 2 in the risk-free test. This result has been found in other experiments as well; the endowment effect appears to be stronger for things than for money. Nevertheless, risk-free and risky loss aversions are definitely related. The correlation across individuals of the two \( \lambda \) measures was 0.635.

There are also other aspects of loss aversion that cannot be assessed by examining simple trade-offs. Loss aversion is particularly problematic in situations where multiple choices and timing are important. Tversky and Kahneman (1981) presented the two choices

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\(^{15}\) If KT utility is assumed with their suggested values of \( \alpha = \beta = 0.88 \), then the mean and median estimates of \( \lambda \) are somewhat lower, but the endowment effect is clearly still present.

\(^{16}\) A small fraction of the participants rejected some lottery and accepted a lottery with a larger loss. These participants’ choices were not used.

\(^{17}\) This table is basically a reproduction of Table 1 in Gächter, Johnson and Herrmann (2010). The numbers here are recomputed from their data to four decimal places to show all the changes. Also they erroneously swapped the results in the last two columns. They determine \( \Omega^{+}(\frac{1}{2}) \) and \( \Omega^{-}(\frac{1}{2}) \) separately; however, a ratio of 0.86 is consistent with a single weighting function satisfying \( \Omega(\frac{1}{2}) = 0.5376 \).
i) Choose between $A: \$240$ or $B:\begin{cases} 
25\% \text{ chance for } \$1000 \\
75\% \text{ chance for } \$0 
\end{cases}$

ii) Choose between $C: -\$750$ or $D:\begin{cases} 
25\% \text{ chance for } \$0 \\
75\% \text{ chance for } -\$1000 
\end{cases}$

to 150 students at Stanford University and the University of British Columbia. The safe choice $A$ was selected over the risky choice $B$ with the higher expected value by 126 students (84%). When only losses were possible, the risky choice $D$ was selected over $C$ by 130 students (87%) even though both had the same expected loss. The modal choice, $A & D$, was made by 110 students (73%). Only 4 (3%) chose $B & C$.

When 86 students were offered the single choice between

Choose between

$A':\begin{cases} 
25\% \text{ chance for } \$240 \\
75\% \text{ chance for } -\$760 
\end{cases}$ or $B':\begin{cases} 
25\% \text{ chance for } \$250 \\
75\% \text{ chance for } -\$750 
\end{cases}$

all chose the first-order stochastically dominating $B'$. Nevertheless, a quick examination will show that $B'$ is just $B$ and $C$ together while $A'$ is a combination of $A$ and $D$, which were preferred individually to $B$ and $C$, respectively.

Though Kahneman and Tversky performed no test to determine why these incompatible choices were made, the simple structure shows it cannot be probability weighting. In each of the four risky gambles there was a 75% chance for a loss and a 25% chance for a gain.¹⁸ Nor is there any need for the loss-aversion kink in utility. The TK utility function with their suggested utility parameters $\alpha = \beta = 0.88$ makes the indicated choices even with $\lambda = 1$. It seems clear that when presented the first choices, the students were evaluating the choice between $A$ and $B$ and between $C$ and $D$ separately. When the gambles were merged into $A'$ and $B'$, evaluations had to be done jointly. There is a distinction between evaluating separate risks and joined risks.

These choices are problematic because they indicate that a small fee would be paid to be able to trade $B$ for $A$ and another small fee would be paid to trade $C$ for $D$, but then a fee would be again paid for trading the joint $A & D$ for $B'$ leaving the decision maker back at the start except for the fees paid. This sequence of trades that leaves the participant strictly worse off is called a Dutch book, and should not be possible in a rational market. This is the reason loss aversion is often called myopic loss aversion. It is applied trade by trade without consideration of future trades.¹⁹

Loss Aversion and the Reference Level

The basic intuition behind loss aversion is that people²⁰ have a stronger preference for avoiding losses than they do for realizing gains. Of course, the very notion of losses and gains requires that there is a reference level from which to measure. The mental process by which

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¹⁸ This is true when a single weighting function is used. It is essentially true if separate weighting functions are applied to gains and losses. If the 0 outcomes in $B$ and $D$ are changed to $-1¢$ and $+1¢$, to make them a loss and gain, respectively, it is exactly true. A small change like that should not alter the choices.

¹⁹ On Wall Street, the phrase myopic risk aversion is usually applied to investors who check their performance too often. Although the overall trend in the stock and other markets is for prices to rise, they do fluctuate and if many small price changes are observed, the losses hurt more than the offsetting gains. This causes the myopically loss averse to hold a smaller position in risky assets.

²⁰ There is evidence that at least some animals, in particular capuchin monkeys, also display loss aversion as reported by Dubner and Levitt in the Freakonomics column, “Monkey Business” New York Times June 5, 2005.
agents set the reference level is called framing. It can depend on how the problem is stated or framed. For example, people seem to assess the two statements: “This costs $50, but you can have a discount of $5 if you pay cash.” and “This costs $45, but there is a $5 surcharge for paying with a credit card.” differently.

To make this explicit in a Finance context, let \( X \) denote the final outcome in consumption or wealth, \( R \) denotes the reference level with \( x \) a profit or loss relative to the reference level. That is, \( X = R + x \). The value function is \( v(x, R) \). Conventionally, the value function is normalized so that \( v(0, R) = 0 \). This value function is increasing in \( x \) and decreasing in the reference level \( R \). Though \( v \) is not concave in \( x \), the function \( V(c, R) \equiv v(c - R, R) \) generally should be concave so there will be an interior optimum.

How is a reference level set? For a simple bet like those here, the status quo seems the obvious reference level. Other researchers have suggested that the expected outcome might serve as the reference level. In investment decisions, the reference level might be determined by the performance alternative assets. For example, when investing \( W \) in a portfolio, the reference level might be \( W(1 + r_f) \), the risk-free alternative. The reference level could also be the return that would have been earned on an alternative investment, \( W(1 + \tilde{r}) \), so the reference level is not only endogenous but random as well. This might be used to evaluate a mutual fund’s performance relative to a stock market index.

What really differentiates loss aversion from risk aversion is that the same ending point is evaluated differently when the reference level changes. If the only evaluation is from the value function, then the same consumption is evaluated differently depending on the reference level. The lower the reference level, the higher the evaluation. If the reference level is fixed exogenously, as has been true this far in our development, this is not an issue. Utility may not be concave, but it is certainly increasing. However, in many models the reference level is endogenous. That can lead to strange predictions because the level of consumption doesn’t matter, only its change with respect to the reference level. If the reference level rises with consumption, the value function could assign lower values even when consumption improved over all.

For example, in a two-period model the reference level for the second period might be first-period consumption. Then the investor would be choosing time-0 consumption and the optimal portfolio return \( \tilde{r} \) to maximize something like \( u(c_0) + v(\tilde{c}_1 - c_0) \) where \( \tilde{c}_1 = (W_0 - c_0)(1 + \tilde{r}) \). One complication with that utility function is that time-1 consumption contributes to utility only to the extent that it exceeds time-0 consumption. Some models amend utility to also include a direct contribution of time-1 consumption. For example, Barberis, Huang, and Santos (2001) posit utility of the form \( u(c_0) + \delta u(\tilde{c}_1) + v(\tilde{c}_1 - c_0) \). The \( u \) function is increasing and concave; the \( v \) function is increasing and S-shaped. In this case the consumption at time 0 reduces the contribution of time-1 utility by decreasing the amount available to invest as well as increasing the reference point.

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21 This simply sets the origin and is permissible for any cardinal utility function. The scaling is still arbitrary.
22 Kőszegi and Rabin (2006) use a utility function of the form \( u(R) + \nu(u(C - u(R)) \) for a reference level of \( R \). Here the argument of the CPT value function is the “profit” or “loss” in utility. Ingersoll and Jin (2012) conjecture a utility function with the multiplicative \( u(R) \cdot \nu(C - R) \).