Chapter 7 — The Portfolio Problem and Pricing

The two previous chapters have considered two special, though widely-used, models of portfolio formation. Both of these models provide useful intuition to how investors decide upon their optimal portfolios and how prices are formed in equilibrium. However, both of them are special and are not necessarily the best description of how investors actually choose portfolios. This chapter looks at a more general model where markets are not complete and the assets have distributions other than elliptical.

The Standard Portfolio Problem

When markets are incomplete, the single-period complete-market portfolio problem facing an investor can be modified by adding a constraint to restrict consumption patterns to those that can be are obtained by trading the available assets.

\[
\max_{c_0, \eta} \sum_s \pi_s U(c_0, c_s; s) \quad \text{subject to} \quad W_0 = c_0 + p'\eta \quad \text{and} \quad c = X\eta.
\]  

As before, \(W_0\) is the investor’s exogenously specified initial wealth,\(^1\) \(c_0\) is the amount consumed at time 0, \(c\) is the vector whose \(s\)th element is the consumption in state \(s\) at time 1. \(X\) is the matrix of payoffs and \(\eta\) is the number of shares of each asset held in the portfolio. For now, the problem is one for an individual investor so the probabilities are his own subjective probabilities. Later it will often be assumed that investors have homogeneous beliefs. Frequently it is assumed that utility is explicitly state-independent and additive over time, \(U(c_0, c_1) = u_0(c_0) + u_1(c_1)\). Often the two utility functions are also assumed to have the same functional form differing only in subjective discounting; that is, \(u_0(c) = u(c), u_1(c) = \delta u(c)\). All of the results below are based on a given individual’s portfolio and consumption choices so the probabilities are those of that investor.

The first constraint is the budget constraint. This was not relevant when examining arbitrage, but now it becomes the primary constraint. Sometimes, particularly in more formal applications, it is given as the inequality, \(W_0 \leq c_0 + p'\eta\). However, whenever utility is strictly increasing, the budget constraint can be expressed as an equality because an investor who prefers more to less will never leave any wealth unconsumed.\(^2\) The second constraint, \(c = X\eta\), is an availability constraint. The assets that are available restrict how consumption can be allocated across the states at time 1.

The problem in (1) is a standard convex programming problem. If a solution exists for a strictly concave \(U\), it is unique. This means that the optimal portfolio is also unique provided there are no redundant assets. From the availability constraint, the portfolio producing the optimal consumption solution is \(c^* = X\eta^*\). When there are no redundant assets, \(X\) has full column rank \(N\) so it has a unique left inverse \((XX)^{-1}\) and the optimal portfolio is \(\eta^* = (XX)^{-1}X'c^*\).

Construct the Lagrangian and determine the first-order conditions for an investor

\[
\begin{align*}
\max_{c_0, \eta} & \quad \mathbb{E}[U(c_0, c; \tilde{s})] + \lambda (W_0 - c_0 - p'\eta) \\
0 & = \frac{\partial \mathcal{L}}{\partial c_0} = \mathbb{E} \left[ \frac{\partial U(\cdot)}{\partial c_0} \right] - \lambda \\
0 & = \mathbb{E} \left[ \frac{\partial U(\cdot)}{\partial c} \right] - \theta \\
0 & = \frac{\partial \mathcal{L}}{\partial \eta} = X'\theta - \lambda p.
\end{align*}
\]  

\(^1\) In an equilibrium model, endowments of assets would be given and determining prices, and thereby initial wealth, would be part of the problem.

\(^2\) As shown in Chapter 5, the budget constraint also might not hold as an equality when it is not possible to construct any portfolios with only nonnegative returns. In such cases free disposal might be optimal.
Substituting for $\lambda$ and $\theta$ from the first two into the third gives

$$\mathbf{p} = \mathbf{X}' \mathbf{q} = \frac{\mathbf{X}' \mathbb{E}[\partial U(\cdot)/\partial c]}{\mathbb{E}[\partial U(\cdot)/\partial c_0]} \equiv \mathbf{X}' \mathbf{q}. \quad (3)$$

The ratio of the investor’s marginal utilities can be defined as his subjective state prices $\mathbf{q} = \theta/\lambda$. Because the market is incomplete, these subjective state prices can differ across the investors even if they have homogeneous beliefs.

In Finance, it is more common to substitute out the availability constraint and present the problem using returns rather than payoffs. The control variables are portfolio weights for each asset rather than the number of shares. Let $\mathbf{r}$ denote the vector of rates of return per dollar invested in the assets and $w$ be the fraction of wealth remaining after time-0 consumption that is invested in each asset. Then the portfolio problem is

$$\max_{c_0, w} \mathbb{E}[U(c_0, (W_0 - c_0)w(1 + \mathbf{r}); s)] \quad \text{subject to} \quad 1' w = 1. \quad (4)$$

For this problem the first-order conditions are

$$0 = \frac{\partial L}{\partial c_0} = \mathbb{E}[U_1(c_0, \mathbf{c}_0; \mathbf{s}) - U_2(c_0, \mathbf{c}_0; \mathbf{s})w(1 + \mathbf{r})]$$

$$0 = \frac{\partial L}{\partial w} = \mathbb{E}[U_2(c_0, \mathbf{c}_0; \mathbf{s})(W_0 - c_0)(1 + \mathbf{r})] - \lambda 1'. \quad (5)$$

$U_1$ and $U_2$ denote the derivatives of $U$ with respect to its first and second arguments. These first-order conditions provide $N+1$ equations for the $N$ asset holdings and time-0 consumption.

Typically, there are no simple expressions for the optimal portfolio as there are when markets are complete or when mean-variance analysis is applicable. One example that can be solved completely is provided in the next section. These models are usually concerned with the pricing relations that can be developed even without solving for the optimal portfolio.

**Pareto Optimal Allocation of Assets**

In a complete-market equilibrium, the allocation of consumption is Pareto optimal. No investor’s expected utility can be increased with a different allocation except by decreasing another investor’s expected utility. This is not true in an incomplete market.

As a very simple example, suppose there is a single risky asset with equally probable returns of $H$ and $L < H$. There are two investors each with one unit of investable wealth and utility functions $u_1(W) = \ell W$ and $u_2(W) = -W^{-1}$. For each investor, the subjective relative state prices are proportional to the marginal utilities so $(q_H/q_L)_1 = L/H$ and $(q_H/q_L)_2 = L^2/H^2$. As $L < H$, investor 2 assigns a higher relative value to consumption in the bad state because he is more risk averse. The subjective state prices are not in the same ratio for both investors so permitting additional trade can increase utility for both.

For example, suppose one unit of consumption in the good state can be swapped for $\theta$ units in the bad state; that is, $\theta$ is the ratio of the price of consumption in the good state relative to the price of consumption in the bad state. If the two investors trade a tiny amount, increasing investor one’s consumption by $\varepsilon$ in the good state while decreasing his consumption by $\theta\varepsilon$ in the bad state, the two utilities would change by

$$\Delta \mathbb{E}[u_1] \bigg|_{\varepsilon = 0^+} = \frac{1}{H} \varepsilon - \frac{1}{L} \theta \varepsilon \quad \text{and} \quad \Delta \mathbb{E}[u_2] \bigg|_{\varepsilon = 0^+} = -\frac{1}{H^2} \varepsilon + \frac{1}{L^2} \theta \varepsilon. \quad (6)$$
Both utilities increase for $\varepsilon > 0$ provided $\theta$ is in the range $L^2/H^2 < \theta < L/H$. That is, a Pareto improvement is possible by trading at any relative price between the two relative subjective prices.

Equation (6) shows that a welfare improving trade is possible; however, the amount that the investors trade in a given market depends on the characteristics of all the investors which together determine the equilibrium value of $\theta^* = (q_H/q_L)^*$. If these two investors act as price takers and are the only investors, both must be satisfied to make no further trades. This condition is met for each investor when the ratio of the marginal utility to the state price is the same in both states for each investor. In other words, the ratios of the marginal utilities in the high and low states must be equal to the relative state prices, $\theta$.

\[
\frac{u'_H(H + \varepsilon)}{u'_L(L - \theta \varepsilon)} = \frac{L - \theta \varepsilon}{H + \varepsilon} = \theta = \frac{u'_H(H - \varepsilon)}{u'_L(L + \theta \varepsilon)} = \frac{(L + \theta \varepsilon)^2}{(H - \varepsilon)^2}.
\]

Solving the left-hand relation for $\varepsilon$ in terms of $\theta$ and substituting that into the right-hand relation leaves a cubic equation in $\theta$. The table below shows some answers as a function of $H$ leaving $L$ fixed at 1.3

<table>
<thead>
<tr>
<th>$H$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>5</th>
<th>10</th>
<th>25</th>
<th>50</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta$</td>
<td>1</td>
<td>0.3984</td>
<td>0.2347</td>
<td>0.1224</td>
<td>0.0522</td>
<td>0.0179</td>
<td>0.0082</td>
<td>0.0039</td>
</tr>
<tr>
<td>$\varepsilon$</td>
<td>0</td>
<td>0.2549</td>
<td>0.6304</td>
<td>1.5863</td>
<td>4.5775</td>
<td>15.4557</td>
<td>35.7361</td>
<td>79.0173</td>
</tr>
<tr>
<td>$\theta \cdot \varepsilon$</td>
<td>0</td>
<td>0.1016</td>
<td>0.1480</td>
<td>0.1941</td>
<td>0.2390</td>
<td>0.2764</td>
<td>0.2942</td>
<td>0.3062</td>
</tr>
</tbody>
</table>

As the disparity between the payoffs grows, which increases the risk in the economy, the more risk averse investor trades more and more ($\varepsilon$) to the more risk tolerant investor in the good state. This buys more “insurance” ($\theta \varepsilon$) but not proportionally more because the value of the payoff in the high state ($\theta$) decreases, increasing the cost of the insurance.

In the just completed example, introducing one additional trading opportunity completed the market and achieved a Pareto optimal allocation. However, it is not always necessary to complete the market to achieve Pareto optimality.

An example is a market a single a risky normally distributed asset, $N(\mu, \sigma)$. Consider two investors with unit wealth, homogeneous beliefs, and exponential utility with risk aversions $a$ and $b > a$. The subjective state prices are proportional to the state probability multiplied by marginal utility so the ratios of subjective state prices in states with gross return $R_1$ and $R_2$ are

\[
\frac{q_a(R_1)}{q_a(R_2)} = \frac{\pi(R_1)}{\pi(R_2)} \exp[-a(R_1 - R_2)] \quad \text{and} \quad \frac{q_b(R_1)}{q_b(R_2)} = \frac{\pi(R_1)}{\pi(R_2)} \exp[-b(R_1 - R_2)].
\]

For $R_2 > R_1$, the more risk averse investor $(b)$ infers a relatively larger subjective state price for state 2 than does investor $a$.

After a risk-free asset is introduced, each investor holds a mix of the risky asset and the safe asset. The portfolio returns are $w_k \tilde{R} + (1 - w_k)R_f$. The ratio of the state prices in states 1 and 2 are then

\[
\frac{q_a(R_1)}{q_a(R_2)} = \frac{\exp(-a[w_k R_1 + (1 - w_k)R_f] + a[w_k R_2 + (1 - w_k)R_f])}{\exp(-b[w_k R_1 + (1 - w_k)R_f] + b[w_k R_2 + (1 - w_k)R_f])} = \frac{\exp[-aw_k(R_1 - R_2)]}{\exp[-bw_k(R_1 - R_2)]}.
\]

\[3 \text{ This describes all solutions. Because both utility functions are CRRA, the state price ratio is homogeneous of degree 0 in } H \text{ and } L \text{ while the optimal trade } \varepsilon \text{ is homogeneous of degree 1 in } H \text{ and } L.\]
As shown in Chapter 5, the optimal portfolio is \( w^*_a = (\mu - r_f)/\sigma^2 \) and similarly for \( w^*_b \). So \( aw_1 = bw_2 \), and the state-price ratio is 1 making the market effectively complete even though there are a continuum of states and only two assets.

In more general economies, completing the market, even effectively completing it, may not be so simple. In this last example if the investors have heterogeneous beliefs, the state price ratio will not be one even after the introduction of a risk-free asset. So the allocation of consumption would not be Pareto optimal. However, the allocation of the assets is always Pareto optimal under the usual modeling assumptions of no trading costs and unrestricted borrowing and short-selling.

To see that the allocation of assets is Pareto optimal, consider the central planner problem of equation (40) in Chapter 4 modified so that the central planner can only make the reallocations permitted by the traded assets. That is, the central planner allocates the assets rather than time-1 consumption. The central planner’s problem is

\[
\max_{c_{1,2}, \ldots, c_k, \eta, \theta} \mathcal{L} = \omega_1 \mathbb{E}^k [U_k(c_k, \bar{c}_h, \tilde{s})] + \sum_{s} \theta_s^0 (\mathbf{X} \eta_s - c_s) + \kappa^0 (\eta^{agg} - \sum_k \eta_s) + \lambda^0 (c_0^{agg} - \sum_k c_k^h).
\]

The \( \theta \)-restrictions guarantee that each state’s allocation can be achieved using only set of traded assets just as in the individual portfolio problem in the previous chapter. The investors’ \( N \) budget constraints are replaced by the \( \kappa \)-restrictions, which ensure that all shares of each of the \( N \) assets are distributed.

The first order conditions are

\[
0 = \omega_k \mathbb{E}^k [\partial U_k / \partial c_k^h] - \lambda, \quad 0 = \omega_k \pi^k \partial U_k / \partial c_{sk} - \theta_{sk}, \quad 0 = \kappa_n - \sum_s \theta_{sk} x_{sk}, \quad \forall k, s, n
\]

Solve the first equation for \( \omega_k \), and substitute it into the second. Then substitute \( \theta_{sk} \) from there into the third leaving

\[
\kappa_n = \frac{\sum_s \pi^s (\partial U_k / \partial c_{sk}) x_{sk}}{\mathbb{E}^k [\partial U_k / \partial c_0^h]} = \lambda \frac{\mathbb{E}^k [(\partial U_k / \partial c_{sk}) \tilde{x}_k]}{\mathbb{E}^k [\partial U_k / \partial c_0^h]}
\]

Different choices for the weights describe different Pareto optimal allocations. The weights need not sum to one, and as the weights are scaled, \( \lambda \) scales with them so it is a free (positive) parameter. Under the competitive equilibrium, \( p_n \) is equal to the utility ratio. So \( \kappa_n = \lambda p_n \) is the Pareto optimal allocations of the assets that is the competitive equilibrium. Nevertheless, the allocation of consumption is not necessarily Pareto optimal. Furthermore, if there are two or more consumption goods, the competitive allocation of assets is not necessarily Pareto optimal.4

### Pricing Relations

To derive the pricing relation, premultiply the \( w \) first-order condition in (5) by \( w' \) giving

\[
\mathbb{E}[U_2(\cdot)(W_0 - c_0)w' (1 + \tilde{r})] = \lambda w' 1 = \lambda.
\]

The left-hand side of this equation is the first term in the \( c_0 \) first-order condition so \( \lambda = (W_0 - c_0) \mathbb{E}[U_1(\cdot)] \). Substituting for \( \lambda \) and canceling \( W_0 - c_0 \) in the second condition leaves the pricing result5

\[
\mathbb{E}[U_2(c_0, \tilde{c}_i, \tilde{s})(1 + \tilde{r})] = \mathbb{E}[U_1(c_0, \tilde{c}_i, \tilde{s})] 1 \quad \text{or} \quad \frac{\mathbb{E}[U_2(c_0, \tilde{c}_i, \tilde{s})(1 + \tilde{r})]}{\mathbb{E}[U_1(c_0, \tilde{c}_i, \tilde{s})]} = 1.
\]

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4 See Diamond (1967) for more details.

5 The relations in (3) and (6) have the same interpretation. With utility written as it is in (4) rather than \( U(c_0, c_1; s) \), time-0 consumption appears in both arguments so \( \partial U_1/\partial c_0 \) applied to (4) would not mean the same as \( U_1 \).
This result can also be derived directly without determining the optimal portfolio. Suppose an investor has determined his optimal time-0 consumption and holds his optimal portfolio. Consider a small reduction in the amount consumed at time 0 that is invested in asset $i$. The change in expected utility is

$$\Delta \mathbb{E}[U] = -\mathbb{E}[U_1(\cdot)]dc_0 + \mathbb{E}[U_2(\cdot)(1 + \tilde{r})]dc_0.$$  \hfill (14)

This change must be zero if the original plan was optimal so (13) holds for every asset.

Equation (13) identifies the SDF as

$$\tilde{m} = \frac{U_2(c_0, \tilde{c}_i ; \tilde{s})}{\mathbb{E}[U_1(c_0, \tilde{c}_i ; \tilde{s})]}.$$  \hfill (15)

This SDF can be applied to gross returns, $\mathbb{E}[\tilde{m}(1 + \tilde{r})] = 1$, or to payoffs, $\mathbb{E}[\tilde{m}x] = p$. As realized marginal utility may differ across investors even with homogeneous beliefs, the individual SDFs can differ.

The pricing result can also be expressed in terms of state prices or a SDF as

$$\sum_s \hat{q}_s(1 + r_s) = 1, \forall i, \text{ where } \hat{q}_s^k \equiv \pi_s^k \frac{U^k(\cdot)}{\mathbb{E}^k[U^k(c_0, \tilde{c}_i ; \tilde{s})]} = \pi_s^k m_s^k.$$  \hfill (16)

Each investor’s state prices are denoted as $\hat{q}_s^k$ because they are subjective, possibly differing across investors when the market is incomplete. However, there are restrictions on the individual state prices. Investors agree on the state-by-state outcomes so the return relations provide $N$ restrictions on each set of subjective state prices. Similarly any portfolio that can be constructed must be priced at 1 as well; $\sum_s \hat{q}_s^k(1 + \sum_i w_ir_i) = 1, \forall w$. For example, if there is an asset or portfolio that has a non-zero gross return $1 + r_s$ in state $s$ and a zero gross return in all other states; that state $s$ is insurable, then from (16)

$$\hat{q}_s^k(1 + r_s) = 1 \Rightarrow \hat{q}_s^k = (1 + r_s)^{-1} = q_s, \forall k.$$  \hfill (17)

All investors must agree on this state price as it is observable, at least indirectly. Investors who have different beliefs about $\pi_s^k$ or different utility functions, adjust their consumption until the equality holds. Similarly if there is a risk-free asset, then $\sum_s \hat{q}_s^k(1 + r_f) = 1$. So all investors’ subjective state prices must sum to the risk-free discount factor, $(1 + r_f)^{-1}$.

If there is a risk-free asset then the interest rate is given by applying (13)

$$\mathbb{E}[U_2(c_0, \tilde{c}_i ; \tilde{s})(1 + r_f)] = \mathbb{E}[U_1(c_0, \tilde{c}_i ; \tilde{s})]1 \Rightarrow 1 + r_f = \frac{\mathbb{E}[U_1(c_0, \tilde{c}_i ; \tilde{s})]}{\mathbb{E}[U_2(c_0, \tilde{c}_i ; \tilde{s})]}.$$  \hfill (18)

In addition, when there is a safe asset, the portfolio problem is often simplified by choosing just the weights for the risky assets, $w$, with $1 - w'1$ invested in the safe asset. Then the portfolio problem is

$$\max_{c_0, w} \mathbb{E}[U(c_0, (W_0 - c_0)[r_f + w'(\tilde{r} - r_f)1]; s)].$$  \hfill (19)

The first-order conditions for the portfolio weights are

$$0 = \frac{\partial \mathcal{L}}{\partial w} = \mathbb{E}[U_2(c_0, \tilde{c}_i ; \tilde{s})(W_0 - c_0)(\tilde{r} - r_f 1)].$$  \hfill (20)

Risk premiums are then given by
\[ 0 = \mathbb{E}[U_2(c_0, \tilde{c}_1; \tilde{s})(\tilde{r} - r_1)] = \text{cov}[U_2(c_0, \tilde{c}_1; \tilde{s}), (\tilde{r} - r_1)] + \mathbb{E}[U_2(c_0, \tilde{c}_1; \tilde{s})]\mathbb{E}[(\tilde{r} - r_1)] \]

\[ \Rightarrow \mathbb{E}[\tilde{r} - r_1] = -\frac{\text{cov}[U_2(c_0, \tilde{c}_1; \tilde{s}), \tilde{r}]}{\text{cov}[U_2(c_0, \tilde{c}_1; \tilde{s})]} = -\frac{\text{cov}[U_2(c_0, \tilde{c}_1; \tilde{s}), \tilde{r}]}{\mathbb{E}[U_1(c_0, \tilde{c}_1; \tilde{s})]}(1 + r_j). \] (21)

The last relation comes from substituting for \( \mathbb{E}[U_2] \) from (13). The expected marginal utility can also be eliminated by using any other asset or portfolio, \( w \), that has a non-zero covariance with marginal utility

\[ \mathbb{E}[\tilde{r} - r_1] = -\frac{\text{cov}[U_2(c_0, \tilde{c}_1; \tilde{s}, \tilde{r})]}{\text{cov}[U_2(c_0, \tilde{c}_1; \tilde{s})]} \mathbb{E}[\tilde{r}_w - r_j]. \] (22)

With time-additive utility, the important relations in (13), (18), (21), and (22) are

\[ 1 = \frac{\mathbb{E}[u'(\tilde{c}_1; \tilde{s})(1 + \tilde{r})]}{u'(c_0)} \quad 1 + r_j = \frac{u'(c_0)}{\mathbb{E}[u'(\tilde{c}_1; \tilde{s})]} = \frac{1}{\mathbb{E}[\tilde{m}]} \]

and

\[ \mathbb{E}[\tilde{r} - r_1] = -\frac{\text{cov}[u'(\tilde{c}_1; \tilde{s}), \tilde{r}]}{u'(c_0)}(1 + r_j) = \frac{\text{cov}[u'(\tilde{c}_1; \tilde{s}), \tilde{r}]}{\text{cov}[u'(\tilde{c}_1; \tilde{s}), \tilde{r}_w]} \mathbb{E}[\tilde{r}_w - r_j]. \] (23)

The discount factor is \((1 + r_j)^{-1} = \mathbb{E}[u'(\tilde{c}_1)]/u'(c_0)\), which is the expected growth in marginal utility.\(^6\) This is not true in general. As shown in (18), the discount factor is the ratio of the expected marginal utilities \( \mathbb{E}[(\partial U/\partial c_1)]/\mathbb{E}[(\partial U/\partial c_0)] \) rather than the expectation of the ratio which is the expected growth \( \mathbb{E}[(\partial U/\partial c_1)/(\partial U/\partial c_0)] \).

There is one important difference between the time-additive result for the risk premium in (23) and the general result in (22). With time-additivity, time-0 utility does not affect risk premiums and time-0 consumption only implicitly affects the risk premium through the amount invested, \( \tilde{c}_1 = (W_0 - c_0)(1 + \tilde{w}\tilde{r}) \).

These results are basically the same with recursive utility, \( \Gamma(c_0, \nu^{-1}(\mathbb{E}[\nu(\tilde{c}_1)])) \). The first order conditions for the optimal portfolio are

\[ 0 = \frac{\partial \Gamma(c_0, \nu^{-1}(\mathbb{E}[\nu(\tilde{c}_1)]))}{\partial \tilde{w}} - \lambda = \Gamma'(c_1) \times \Lambda'(\gamma) \times \mathbb{E}[\nu'(\tilde{c}_1)(W_0 - c_0)(1 + \tilde{r})] - \lambda \mathbb{E}[\nu'(\tilde{c}_1)(1 + \tilde{r})]. \] (24)

where \( \Lambda(\cdot) = \nu^{-1}(\cdot) \) denotes the inverse function. The second line follows because the canceled factors are the same for all assets and can be absorbed into the new Lagrange multiplier, \( \tilde{\lambda} \). This means that \( \mathbb{E}[\nu'(\tilde{c}_1)(\tilde{r}_1 - \tilde{r}_2)] = 0 \) for all pairs of assets. In particular, if there is a risk-free asset, then \( \mathbb{E}[\nu'(\tilde{c}_1)(\tilde{r}_1 - \tilde{r}_2)] = 0 \). The same algebra as in (21) gives the final result in (23)

\[ \mathbb{E}[\tilde{r} - r_1] = -\frac{\text{cov}[\nu'(\tilde{c}_1; \tilde{s}), \tilde{r}]}{\text{cov}[\nu'(\tilde{c}_1; \tilde{s}), \tilde{r}] \mathbb{E}[\tilde{r}_w - r_j]} \mathbb{E}[\tilde{r}_w - r_j]. \] (25)

The first-order condition for optimal time-0 consumption is

\[ 0 = \frac{\partial \Gamma(c_0, \nu^{-1}(\mathbb{E}[\nu(\tilde{c}_1)]))}{\partial c_0} = \Gamma'(c_1) + \Gamma'(\gamma) \times \Lambda'(\mathbb{E}[\nu(\tilde{c}_1)]) \times \mathbb{E}[-\nu'(\tilde{c}_1)(1 + \tilde{r})]. \] (26)

Note that the differentiation with respect to \( c_0 \) here is not the same as \( U_1 \) holding \( c_1 \) fixed. Rather the substitution \( \tilde{c}_1 = (W_0 - c_0)(1 + \tilde{r}^*) \) is used. From (24), \( \mathbb{E}[\nu'(\tilde{c}_1)(1 + \tilde{r}^*)] = \mathbb{E}[\nu'(\tilde{c}_1)(1 + r_j)] \)

\(^6\) The discount factor is not the expected growth in marginal utility in the absence of time-additivity. The expected growth in marginal utility is \( \mathbb{E}[(\partial U/\partial c_1)/(\partial U/\partial c_0)] \).
because this expectation is the same for all assets and, therefore, for all portfolios. Substituting into (26) and solving gives

\[1 + r_f = \frac{\Gamma_1(\cdot)}{\Gamma_2(\cdot) \times \Lambda'(\mathbb{E}[v(c_1)]) \times \mathbb{E}[v'(c_1)]}.\]  

(27)

For EZ preferences, risk utility is \(v(c_1) = c_1^{1-\gamma}\), so

\[\mathbb{E}[\bar{c}_1^{-\gamma}(1 + \bar{r})] = \bar{\lambda} 1, \quad \text{and} \quad \mathbb{E}[\bar{r} - r_f] = \frac{\text{cov}[(1 + \bar{r})^{-\gamma}, \bar{r}]}{\text{cov}[(1 + \bar{r})^{-\gamma}, \bar{r}_w]} \mathbb{E}[\bar{r}_w - r_f].\]  

(28)

where \(\bar{r}\) is the rate of return on the optimally invested portfolio. This simplification is one reason that EZ preferences including CRRA utility are so commonly used in models.

The aggregator is \(\Gamma(c_0, \psi_1) = (c_0^\rho + \delta \psi_0^\rho) / \rho, \quad \psi_1 = (\mathbb{E}[\bar{c}_1^{-\gamma}])^{1/(1-\gamma)}\) with the inverse utility function is \(\Lambda(\theta) = \theta^{1/(1-\gamma)}\) and its derivative \(\Lambda'(\theta) = (1 - \gamma) \theta^{-(1/(1-\gamma))}\). So, from (27) the interest rate is

\[1 + r_f = \frac{c_0^{\rho-1}}{\delta(\mathbb{E}[\bar{c}_1^{-\gamma}])^{(\rho-1)/(1-\gamma)} \times (\mathbb{E}[\bar{c}_1^{-\gamma}])^{1/(1-\gamma) \times \mathbb{E}[\bar{c}_1^{-\gamma}]}} = \frac{c_0^{\rho-1}}{\delta(\mathbb{E}[\bar{c}_1^{-\gamma}])^{(\rho+\gamma-1)/(1-\gamma)} \times \mathbb{E}[\bar{c}_1^{-\gamma}]} .\]  

(29)

Time-additive CRRA utility is the special case with \(\rho = 1 - \gamma\) so

\[1 + r_f = c_0^{\gamma-1} / \delta \mathbb{E}[\bar{c}_1^{-\gamma}].\]  

(30)

Some models, like the CAPM, are time-additive by default by not mentioning time-zero consumption at all. In these models, the interest rate obviously cannot be determined by the growth rate of the marginal utility of consumption. Of course, there still can be a risk-free asset (or portfolio) if there is one with a constant payoff across states. But with no time-0 consumption, only relative asset prices are determined. These models are typically cleared in one of two fashions. One way is to simply state an interest rate which serves to define the numeraire for time-0 wealth. The other way is to ignore time-0 wealth and give investors endowments of time-1 wealth or consumption. Often, in such models, outcomes and utility functions are defined in terms of wealth as there is no time-0 consumption for comparison. The relations for excess returns as in (21) or (22) are

\[\mathbb{E}[(\bar{r} - r_f)\mathbf{1}] = -\frac{\text{cov}[u'(\bar{W}; \bar{s}), \bar{r}]}{\mathbb{E}[u'(\bar{W}; \bar{s})]} = \frac{\text{cov}[u'(\bar{W}; \bar{s}), \bar{r}]}{\text{cov}[u'(\bar{W}; \bar{s}), \bar{r}_w]} \mathbb{E}[\bar{r}_w - r_f].\]  

(31)

where \(\mathbf{w}\) is any portfolio with a non-zero correlation with marginal utility.

The results here are not equilibrium effects. They are purely individual investor results. They hold for any investor’s utility along with his or her subjective probability beliefs. Many models in finance consist of determining exactly what that utility is. In some models like the CAPM, we can aggregate demands to relate risk premiums to the market portfolio. Another increasingly more common approach is to assume a representative investor with a known utility function who holds the market portfolio.

There are a few complications with the latter approach. First, it is possible to construct

\textit{Footnote:} For EZ preferences, \(v(c) = c^{1/\gamma}\) can be used in place of the usual \(u(c) = c^{1/\gamma}/(1 - \gamma)\) even when \(\gamma > 1\). The derivative \(v'\) is negative, but taking the \((1 - \gamma)th\) root ensures that the certainty equivalent, \(\psi_1\), is increasing in \(c_1\). Similarly, using \(\rho\) as a divisor rather than a root in the aggregator is permissible in a single-period problem as \(\psi_0\) need not be a certainty equivalent.
economies in which all investors are risk averse, but the market portfolio is not optimal for any risk averse investor; that is, a representative investor who is average simply does not exist. The next chapter presents an example. Second in multi-period models, the standard representative investor is average both in his portfolio and consumption each period; that is, he holds the market portfolio and consumes the per capita amount. In a single-period model the second property follows directly from the first as all wealth is consumed. In a multi-period model, these two concepts are distinct, and it may be difficult to ensure that both conditions are met.

State Prices in Incomplete Markets

Equations like (16) determine the SDF or state prices for every state; however, in many models this is more than necessary. If an optimal portfolio and the associated utility function is known, perhaps by the assumption of a representative investor who holds the market portfolio, then pricing can be done by just focusing on the states with distinct payoffs on the optimal portfolio.

The first step is to group the states with the same consumption into aggregate states. A generic aggregate state is denoted by $A$. For states within each aggregate state consumption and therefore marginal utility are constant so the state prices satisfy

$$q_s = \pi_s \frac{u'(c_s)}{u'_0(c_0)} = \pi_s \frac{u'(c_A)}{u'_0(c_0)} = \pi_{s,t} \times \pi_t \frac{u'(c_A)}{u'_0(c_0)} \equiv \pi_{s,t} Q_A$$

where the aggregate state price is defined as $Q_A \equiv \pi_{s,t} u'(c_A) / u'_0(c_0)$. For non-time-additive utility, the aggregate state price is defined using (21).

The standard pricing result can be given in terms of these aggregate state prices as

$$p_i = \sum_s q_s X_{si} = \sum_A Q_A \sum_{s \in A} \pi_{s,t} X_{si} = \sum_A Q_A \mathbb{E}[\bar{x}_i | A] \equiv \sum_A Q_A \bar{x}_i (A).$$

So to value any asset we need to know its conditional expected payoff in each aggregate state and the aggregate-state prices. This result can also be stated in terms of a SDF by expressing the aggregate-state price as $Q_A = \pi_t M_A$,

$$p_i = \sum_A Q_A \bar{x}_i (A) = \sum_A \pi_t M_A \bar{x}_i (A) = \mathbb{E}_i [\tilde{M} \cdot \tilde{x}_i].$$

This is an individual result based on one investor’s consumption. As shown in the next chapter, the market portfolio is not necessarily efficient even if investors have homogeneous beliefs so the aggregation is not necessarily passed on the market portfolio. However, if the market is effectively complete, then investors with homogeneous beliefs do have their portfolios aligned. They must agree on the prices for each aggregate-state that can be distinguished by a different outcome on the market portfolio, and the SDF can be generated from the market portfolio's return.

The Normal-Exponential Model and the CAPM Once More

If an investor with an exponential utility function has consumption that is normally distribution, the pricing relations are relatively simple. From (23) and the moment generating function for a normal
\[
1 + r_f = \frac{u'(\tilde{c}_0)}{\E[u'(c_0)]} = \delta^{-1} \exp(-ac_0) \frac{\delta^{-1} \exp(-ac_0)}{\E[\exp(-a\tilde{c}_1)] = \exp(-a\E[\tilde{c}_1] + \frac{1}{2}a^2 \text{var}[\tilde{c}_1])} = \delta^{-1} \exp(-ac_0 + a\E[\tilde{c}_1] - \frac{1}{2}a^2 \text{var}[\tilde{c}_1])
\]

\[\Rightarrow r_{\text{cont-com}} \equiv \ln(1 + r_f) = -\ln \delta + a\E[\tilde{c}_1 - c_0] - \frac{1}{2}a^2 \text{var}[\tilde{c}_1].\]  

The risk premium on any asset can also be determined from (23) using Stein’s Lemma which states that if \(\hat{x}\) and \(\hat{z}\) have a joint normal distribution, and \(h\) is a differentiable function satisfying \(\E[|h(\hat{x})|] < \infty\), then \(\text{cov}[h(\hat{x}), \hat{z}] = \text{cov}[\hat{x}, \hat{z}] \cdot \E[h'(\hat{x})].\)  

\[
\E[\tilde{r} - r_f \mathbf{1}] = -\frac{\text{cov}[u'(\tilde{c}_1), \tilde{r}]}{u'(c_0)}(1 + r_f) = -\frac{\text{cov}(\tilde{c}_1, \tilde{r}) \E[u'(\tilde{c}_1)]}{u'(c_0)}(1 + r_f)
\]

\[
= -\frac{\text{cov}(\tilde{c}_1, \tilde{r}) - \delta a \exp(-a\E[\tilde{c}_1] + \frac{1}{2}a^2 \text{var}[\tilde{c}_1])}{\exp(-ac_0)}(1 + r_f) = a \text{cov}(\tilde{c}_1, \tilde{r}).
\]

As this applies to any asset with a joint normal distribution with consumption, \(a\) can be solved for using any asset with a non-zero risk premium and

\[
\E[\tilde{r} - r_f \mathbf{1}] = a \text{cov}(\tilde{c}_1, \tilde{r}) = \frac{\text{cov}(\tilde{c}_1, \tilde{r})}{\text{cov}(\tilde{c}_1, \tilde{r}^*)} \E[\tilde{r}^* - r_f].
\]  

Furthermore, for any investor \(k\), his optimal portfolio has returns that are perfectly correlated with his consumption

\[
\E[\tilde{r} - r_f \mathbf{1}] = \frac{\text{cov}(\tilde{c}_1, \tilde{r})}{\text{var}(\tilde{r}^*)} \E[\tilde{r}^* - r_f],
\]

which is exactly the CAPM result except that \(\tilde{r}^*\) is not necessarily the market portfolio.

If only some assets have returns with a joint normal distribution with consumption, then this relation applies to them. So this result extends that in previous chapter. The CAPM can apply to a subset of the assets in positive supply. This result can also be extended to price derivative contracts written on assets with normal distributions as shown later.

### The Lognormal-CRRA Model

Another pair of assumptions commonly used in portfolio models are lognormally distributed consumption and asset prices and CRRA utility or EZ preferences. These are convenient joint assumptions because when \(\tilde{\omega}\) is lognormally distributed, \(\tilde{w} \equiv \ln \tilde{\omega}\) is normally distributed, and CRRA utility or marginal utility is \(\omega^n = e^{\alpha/\sigma^2} = e^{\omega}\) so the normal moment generating function can be employed again.

\[
\E[\tilde{\omega}^n] = \E[\exp(\alpha \tilde{w})] = \exp(\alpha \tilde{w} + \frac{1}{2} \alpha^2 \sigma^2) = \tilde{\omega}^n \exp[\frac{1}{2}(\alpha^2 - \alpha)\sigma^2].
\]

The last equality follows because \(\tilde{\omega} = \exp(\tilde{w} + \frac{1}{2}\sigma^2)\). Each of these two expression are useful.

Applying the first expression in (39) to (29), the risk-free rate is

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8 This result also holds for joint elliptically distributed variables. As shown by Landsman (2006), Stein’s Lemma can be extended to joint elliptical variables as \(\text{cov}[h(\hat{x}), \hat{z}] = \text{cov}(\hat{x}, \hat{z}) \cdot \E[h'(\hat{x})]\). The random variable \(\hat{x}\) is one associated with the same elliptical family but with dimension one smaller. \(\E\) is the expectation under this latter distribution. The expectations of the derivative of \(h\) would appear in both the numerator and denominator of (27) and can be canceled so the exact form is irrelevant.
\[1 + r_j = \frac{c_0^{p-1}}{\delta (E[c_1^{-\gamma}])^\frac{(p+\gamma-1)(1-\gamma)}{\gamma}} \times E[c_1] \]
\[= \delta^{-1} c_0^{p-1} \exp \left( -\frac{\rho + \gamma - 1}{1-\gamma} \left[ (1-\gamma)E[\ell n c_1] + \frac{1}{2} (1-\gamma^2) \text{var}(\ell n c_1) \right] + \gamma E[\ell n c_1] - \frac{1}{2} \gamma^2 \text{var}(\ell n c_1) \right) \tag{40} \]
\[= \delta^{-1} c_0^{p-1} \exp \left( E[\ell n \tilde{c}_1] (1-\rho) - \frac{1}{2} \gamma \text{var}(\ell n \tilde{c}_1) [\rho (1-\gamma) + 2\gamma - 1] \right). \]

From (39), \( E[\ell n \tilde{c}_1] = \ell n (\bar{c}_1) - \frac{1}{2} \text{var}(\ell n \tilde{c}_1) \) so
\[1 + r_j = \delta^{-1} c_0^{p-1} \exp \left( (\ell n (\bar{c}_1) - \frac{1}{2} \text{var}(\ell n \tilde{c}_1))(1-\rho) - \frac{1}{2} \text{var}(\ell n \tilde{c}_1) [\rho (1-\gamma) + 2\gamma - 1] \right) \]
\[= \delta^{-1} c_0^{p-1} \exp \left( -\frac{1}{2} \gamma (2-\rho) \text{var}(\ell n \tilde{c}_1) \right) \tag{41} \]

For the continuously-compounded rate
\[r_{\text{cont-com}} \equiv \ell n (1 + r_j) = -\ell n \delta + (1-\rho) E[\ell n \tilde{c}_1 - \ell n c_0] - \frac{1}{2} [\rho (1-\gamma) + 2\gamma - 1] \text{var}(\ell n \tilde{c}_1) \]
and \( r_{\text{cont-com}} = -\ell n \delta + (1-\rho) (\ell n \bar{c}_1/c_0) - \frac{1}{2} \gamma (2-\rho) \text{var}(\ell n \tilde{c}_1). \tag{42} \]

The first term is the continuously-compounded subjective rate of time. A smaller \( \delta \) means that the investor values current consumption more relative to future consumption. The more impatient is the investor, the larger must be the interest rate to induce the savings required to achieve the same amount of time-1 consumption. The interest rate must also be larger when the growth rate in consumption is higher, and there is more time-1 consumption on average. As the investor has decreasing marginal utility, he would like to borrow more to finance time-0 consumption thereby smoothing total consumption more. A higher interest rate makes this borrowing unattractive. For CRRA utility, the factor \( 1-\rho \) in this term would be \( \gamma \), the risk aversion, but it is clear for EZ preferences that it is not risk aversion, but the elasticity of intertemporal consumption that is important. This should be clear because this term does not depend on the variance so risk aversion cannot be relevant. The final term is the risk term. This portion of the interest rate results from the precautionary demand for savings. When the risk of time-1 consumption is high, risk-averse consumers would move their investments into the safe asset; a lower interest rate makes this less desirable and restores the equilibrium. Both risk aversion and the elasticity have roles in the precautionary demand. The investor wishes to smooth his consumption both across states and time.

If the gross return on some asset is joint lognormally distributed with consumption, then (13) can be applied in a similar fashion. The logarithm of the random variable in the numerator, \(-\gamma \ell n \tilde{c}_1 + \ell n (1+\tilde{r})\), is normally distributed and, using (39),
\[1 = \delta c_0^{\gamma} E[c_1^{-\gamma} (1+\tilde{r})] = \delta c_0^{\gamma} E \left[ \exp \left( -\gamma \ell n \tilde{c}_1 + \ell n (1+\tilde{r}) \right) \right] \]
\[= \delta c_0^{\gamma} \exp \left( -\gamma E[\ell n \tilde{c}_1] + E[\ell n (1+\tilde{r})] + \frac{1}{2} \text{var}[\gamma \ell n \tilde{c}_1 + \ell n (1+\tilde{r})] \right) \]
\[= \exp \left[ \ell n (1+\tilde{r}) \right] = -\ell n \delta - \gamma \ell n c_0 + \gamma E[\ell n \tilde{c}_1] - \frac{1}{2} \text{var}[\gamma \ell n \tilde{c}_1 + \ell n (1+\tilde{r})] \]. \tag{43} \]

The same is true for the risk-free asset, therefore
\[E[\ell n (1+\tilde{r})] - \ell n (1+ r_j) = -\frac{1}{2} \text{var}[\gamma \ell n \tilde{c}_1 + \ell n (1+\tilde{r})] + \frac{1}{2} \text{var}[\gamma \ell n \tilde{c}_1] \]
\[= \gamma \text{cov}[\ell n \tilde{c}_1, \ell n (1+\tilde{r})] - \frac{1}{2} \text{var}[\ell n (1+\tilde{r})]. \tag{44} \]

The relation is exactly the same for recursive utility with a CRRA risk assessment. However, this is not true in a multi-period model. In multi-period models, the EISC also affects the risk premium because the end-of-period consumption generally differs between recursive and additive utility.
The lognormal-CRRA (or EZ) model gives simple intuitive results much like the CAPM. Apart from a Jensen’s inequality correction, expected log returns are explained by a single covariance along with a utility parameter. The utility parameter can be eliminated as in (28) by introducing any portfolio with a non-zero covariance with consumption

$$\mathbb{E}[\ln(1+r_i)] - \ln(n(1+r_f)) - \frac{1}{2} \text{var}[\ln(n(1+r_f))]$$

$$= \frac{\text{cov}[\ln\tilde{c}_i, \ln(1+r_i)]}{\text{cov}[\ln\tilde{c}_i, \ln(1+r_f)]} \left( \mathbb{E}[\ln(1+r_f)] - \ln(n(1+r_f)) - \frac{1}{2} \text{var}[\ln(n(1+r_f))] \right).$$  \hspace{1cm} (45)

### The Valuation of Derivative Claims

A useful application of the joint preference-distribution models is the valuation of derivative claims like options. Then SDF for an investor with time-additive utility is the ratio of time-1 marginal utility to time-0 marginal utility. Therefore, for a derivative asset written on some underlying basis, $\tilde{z}$, with a payoff of $h(\tilde{z})$, the present value is

$$\text{PV}[h(\tilde{z})] = \mathbb{E}[h(\tilde{z})u'(\tilde{c}_1)]/u'_0(c_0).$$  \hspace{1cm} (46)

To specialize this valuation, assume that time-1 consumption and $\tilde{z}$ have a joint normal distribution, $N(\mu_c, \mu_z, \sigma_z^2, \sigma_c^2, \rho)$, and that utility is $-(e^{-\delta}e^{-\delta})/a$. The value of any derivative contract whose payoff depends only on $\tilde{z}$ and not directly on consumption is

$$C = \text{PV}[h(\tilde{z})] = \mathbb{E}[\tilde{m}h(\tilde{z})] = e^{ac_0} \delta \mathbb{E}[e^{-a\tilde{z}}h(\tilde{z})] = e^{ac_0} \delta \mathbb{E}[h(\tilde{z})|z]]$$  \hspace{1cm} (47)

The inner expectation is moment generating function of the distribution of time-1 consumption conditional on a given $z$. As the two have a joint normal distribution, the conditional distribution of $\tilde{c}_1$ is normal with an expectation $\mu_{c|z} = \mu_c + \rho \sigma_c (z - \mu_z)/\sigma_z$ and a variance $(1 - \rho^2)\sigma_c^2$. So the inner expectation is

$$\mathbb{E}[e^{-a\tilde{z}^2}|z] = \exp\left(-a\mu_{c|z} + \frac{1}{2}a^2(1 - \rho^2)\sigma_z^2\right).$$  \hspace{1cm} (48)

Substituting (48), $\mu(c)$, and the normal density function for $z$ into (47), the derivative contract’s present value is

$$C = \frac{\delta \exp(ac_0 + \frac{1}{2}a^2(1 - \rho^2)\sigma_z^2)}{\sqrt{2\pi}\sigma_z} \int_{-\infty}^{\infty} h(z) \exp\left[-a\mu_{c|z} - (z - \mu_z)^2/2\sigma_z^2\right] dz.$$  \hspace{1cm} (49)

The exponent in the integral can be rewritten as

$$-a\mu_{c|z} - (z - \mu_z)^2/2\sigma_z^2 = -a\mu_c - \frac{2a\rho \sigma_c (z - \mu_z) + (z - \mu_z)^2}{2\sigma_z^2}$$

$$= -a\mu_c + a^2\rho^2\sigma_z^2/2 - (z - \mu_z + a\rho \sigma_c \sigma_z)^2/2\sigma_z^2.$$  \hspace{1cm} (50)

Combining with (49), the derivative contract’s value is

$$C = \frac{\delta \exp[a(c_0 - \mu_c) + \frac{1}{2}a^2\sigma_z^2]}{\sqrt{2\pi}\sigma_z} \int_{-\infty}^{\infty} h(z) \exp\left[-(z - \mu_z + a\rho \sigma_c \sigma_z)^2/2\sigma_z^2\right] dz.$$  \hspace{1cm} (51)

From (35), the numerator of the lead factor is the risk-free discount factor. The exponential in the integral along with the denominator of the lead factor is the normal density function so

$$C = (1+r_f)^{-\frac{1}{2}\delta\mathbb{E}[h(z)]}.$$  \hspace{1cm} (52)
The hatted expectation indicates it is the risk-neutral expectation of $\tilde{z}$ being used. The mean has been altered to $\hat{\mu}_z = \mu_z - a \rho \sigma_z \sigma_c$, and the variance is unchanged.

Suppose now that $\tilde{z}$ is time-1 value of an asset, $\tilde{p}_0$, whose payoff at time 1 is $\tilde{p}_1$, and $h$ is the identity function. This asset is usually called the underlying asset, the basis asset, or simply the basis. The discounted value in (52) must give the current price of the basis so

$$p_0 = (1 + r_f)^{-1} \tilde{E}[\tilde{p}_1] = (1 + r_f)^{-1} \hat{\mu}_p.$$  \hspace{1cm} (53)

The risk-neutral mean of $\tilde{p}_1$ is identified as $\hat{\mu}_p = (1 + r_f)p_0$. Of course, this must be true without deriving it, the risk-neutral expected rate of return must be equal to the interest rate on all assets. Nevertheless, it is comforting to confirm that it is correct for this problem.

Combining (53) and (52), the valuation in (51) of the derivative can be written as

$$C = \frac{1}{(1 + r_f)} \frac{1}{\sqrt{2\pi\sigma_p}} \int_{-\infty}^{\infty} h(p_1) \exp\left(-\left[p_1 - p_0(1 + r_f)\right]^2 / 2\sigma_p^2\right) dp_1. \hspace{1cm} (54)$$

To use (54), the only parameters we need to know are the interest rate and the standard deviation of the time-1 price. The risk aversion parameter has been eliminated. Nor is there any need to know the appropriate discount rate for the derivative or the true expected rate of return on the underlying asset. This must seem like magic. How can the value of a derivative contract not depend on its discount rate or the expected rate of return of the basis? The answer is they do, but they do so only through the price and the interest rate.\(^9\)

From (35), risk aversion affects the interest rate. If any change in $a$ is to leave the interest rate unaffected, then distribution of time-1 consumption must change. Of course, in any application that uses (54), that change is irrelevant. The change in the time-1 consumption also would affect the price of the underlying. It that is to remain the same, then its expectation or its correlation with consumption must change, but again these do not affect the use of (54). The two changes offset. This must seem like a great coincidence, but it really is not. We know that risk-neutral valuation is valid in the absence of arbitrage. The only real coincidence here is that the risk-neutral distribution is just like the true distribution with a simple change in the expectation.

To summarize, when an asset price and consumption have a joint normal distribution and utility is exponential, the value of a derivative contract written on the asset price is determined by changing the expected rate of return on the underlying asset to the interest rate and discounting the derivative’s payoff at the interest rate.

A similar result obtains if utility is power in form, $c^{1-\gamma}/(1-\gamma)$, and consumption and the price have a joint lognormal distribution so the logs of consumption and the price have a joint normal distribution. The parameters are $E[\ln p_1] \equiv \mu_{ln p}$, $E[\ln c_1] \equiv \mu_{ln c}$, $\text{var}[\ln p_1] \equiv \sigma_{ln p}^2$, $\text{var}[\ln c_1] \equiv \sigma_{ln c}^2$, and $\text{cov}[\ln p_1, \ln c_1] \equiv \rho \sigma_{ln p} \sigma_{ln c}$. Because $\tilde{p}_1$ has a lognormal distribution

$$E[\tilde{p}_1] = E[e^{\mu_{ln \tilde{p}}}] = \exp(\mu_{ln p} + \frac{1}{2} \sigma_{ln p}^2). \hspace{1cm} (55)$$

The same analysis applies as before using logs. The value of any derivative contract is

$$C = \text{PV}[h(\ln \tilde{p}_1)] = c_0^{-\gamma} E[\delta c_1^{-\gamma} h(\ln \tilde{p}_1)] = \delta c_0^{-\gamma} E[h(\ln \tilde{p}_1)]E[c_1^{-\gamma} \mid p_1] = \delta e^{-\gamma \ln \rho} E[h(\ln \tilde{p}_1)]E[e^{-\gamma \ln c_1} \mid p_1]. \hspace{1cm} (56)$$

Comparing (56) to (47), it is immediate that

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\(^9\) The is one caveat in the application of (44). If the derivative contract is not already traded in the market, then introducing it can affect the equilibrium and change the interest rate or the price of the basis. The equation must be applied using the equilibrium $r_f$ and $p_0$ observed after the introduction of the derivative contract.
\[ C = (1 + r_f)^{-1} \hat{E}_p[h(\ell n \tilde{p}_1)] \quad \text{where} \quad \hat{\mu}_{in,p} = \mu_{in,p} - \gamma \rho \sigma_{in,p} \sigma_{in,c}. \quad (57) \]

If \( h(\ell n p_1) = \exp(\ell n p_1) = p_1 \), then \( C \) is the present value of the underlying asset so

\[
p_0 = (1 + r_f)^{-1} \hat{E}_p[e^{\ell n \tilde{p}_1}] = (1 + r_f)^{-1} \exp(\hat{\mu}_{in,p} + \frac{1}{2} \sigma_{in,p}^2) \]

\[\Rightarrow \hat{\mu}_{in,p} = \ell n[p_0(1 + r_f)] - \frac{1}{2} \sigma_{in,p}^2 \quad (58)\]

\[\Rightarrow \hat{E}_p[\tilde{p}_1] = \hat{E}_p[e^{\ell n \tilde{p}_1}] = \exp(\hat{\mu}_{in,p} + \frac{1}{2} \sigma_{in,p}^2) = \exp(\ell n[p_0(1 + r_f)]) = p_0(1 + r_f). \]

So once again the risk-neutral distribution is the natural distribution with expected rate of return on the asset replaced by the interest rate and with the (logarithmic) variance unchanged.

For a call option with a payoff of \( \max(p_1 - k, 0) \), taking the risk-neutral expectation gives

\[ C = p_0 \Phi(Z^-) - (1 + r_f)^{-1} k \Phi(Z^-) \quad \text{where} \quad Z^\pm \equiv \frac{\ell n(p_0(1 + r_f)/k) \pm \frac{1}{2} \sigma_{in,p}^2}{\sigma_{in,p}} \quad (59) \]

This is the well-known Black-Scholes option pricing formula. It was originally, and is usually, derived by no-arbitrage arguments in a model where the stock price evolves according to a lognormal diffusion process. That development will be seen in a later chapter. This derivation is based on Rubinstein (1976) and Brennan (1979). Again the risk aversion parameter has disappeared from the formula so (59) provides a simple risk-neutral valuation.

This result seems to mimic the first. The risk aversion, expected rate of return on the basis, and the discount rate for the derivative are not required. A risk-neutral evaluation is used and the only change in the distribution is an adjustment to the basis’ expected rate of return. However, this is not quite correct. A different mean is the only change in the distribution of the log of the price, but the variance of the raw price also changes. The variance of the lognormal distribution of the price is \( \text{var}[\tilde{p}_1] = \mu_p^2 \exp(\sigma_{in,p}^2)[\exp(\sigma_{in,p}^2) - 1] \). The risk-neutral variance has the same formula with \( \mu_p \) replaced by \( 1 + r_f \), so \( \text{var}[\tilde{p}_1] = \exp[\ell n p_0(1 + r_f)] \). So \( \text{var}[\tilde{p}_1] = \exp[\ell n p_0(1 + r_f)] \).

The CARA-normal and CRRA-lognormal models are popular because the risk-neutral evaluation appears so much like an expected discounted value under the natural probabilities. Unfortunately, risk-neutral valuation, while always valid when there is no arbitrage, is not so simple with other distribution and utility combinations. The risk-neutral distribution often differs not only in parameters, but has a completely different functional form as well.

As a simple illustration suppose consumption has a Laplace distribution with density and cumulative

\[ f(c_1) = \frac{1}{2} \beta \exp(-\beta |c_1 - \mu|) \quad -\infty < c_1 < \infty \]

\[ F(c_1) = \begin{cases} \frac{1}{2} \exp[\beta(c_1 - \mu)] & c_1 < \mu \\ 1 - \frac{1}{2} \exp[-\beta(c_1 - \mu)] & c_1 > \mu. \end{cases} \quad (60) \]

The mean and variance of the distribution are \( \mathbb{E}[c_1] = \mu \) and \( \text{var}[c_1] = 2/\beta^2 \).

For exponential utility with \( a < \beta \), the discount factor is

\[ \frac{1}{1 + r_f} = e^{\alpha c_0} \mathbb{E}[\delta e^{-ac_0}] = \frac{1}{2} \delta e^{\alpha c_0} \beta \int_{-\infty}^{\infty} e^{-ae^{-\beta(x - \mu)}} dx = \delta \frac{\exp[a(c_0 - \mu)]}{1 - a^2/\beta^2}. \quad (61) \]

The value of a derivative claim with payoff, \( h(c_1) \), is

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10 If \( a > \beta \), then expected utility is \( -\infty \) because utility becomes unboundedly negative faster than the probability goes to zero.
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\[ \text{PV}[h(\tilde{c}_i)] = \delta e^{\alpha_0} \mathbb{E}[h(\tilde{c}_i)e^{-\alpha_1}] = \frac{1}{2} \delta e^{\alpha_0} \beta \int_{-\infty}^{\infty} h(c) e^{-ac - \beta(c - \mu)} dc \]

\[ = \frac{1}{1 + r_f} \frac{\beta^2 - a^2}{2\beta} \int_{-\infty}^{k} h(c) e^{-a(c - \mu) - \beta(c - \mu)} dc . \]  

(62)

The second equality follows after substitution of (61)

To explore further, consider pricing binary or digital put option with strike price \( k \) on consumption. This contract pays \( 1 \) if \( \tilde{c}_1 < k \) and \( 0 \) otherwise. Using (62), the value of this contract is

\[ \text{PV}[1_{\tilde{c}_1 < k}] = \frac{1}{1 + r_f} \frac{\beta^2 - a^2}{2\beta} \int_{-\infty}^{k} e^{-a(c - \mu) - \beta(c - \mu)} dc = \frac{1}{1 + r_f} \left\{ \begin{array}{ll} \frac{\beta + a}{2\beta} e^{(\beta + a)(k - \mu)} & k \leq \mu \\
1 - \frac{\beta - a}{2\beta} e^{-(\beta + a)(k - \mu)} & \mu \geq k . \end{array} \right. \]  

(63)

Equation (63) also gives the risk-neutral cumulative probability distribution once the discount factor is excluded because \( \text{PV}[1_{\tilde{c}_1 < k}] = (1 + r_f)^{-1} \mathbb{P}(1_{\tilde{c}_1 < k}) \). The risk-neutral probability density is the derivative

\[ \hat{f}(c_1) = \frac{\partial}{\partial k} \left( (1 + r_f) \text{PV}[1_{\tilde{c}_1 < k}] \right) = \left\{ \begin{array}{ll} \frac{\beta^2 - a^2}{2\beta} e^{(\beta - a)(k - \mu)} & k < \mu \\
\frac{\beta^2 - a^2}{2\beta} e^{-(\beta + a)(k - \mu)} & \mu > k . \end{array} \right. \]  

(64)

Therefore, the valuation in (62) for any derivative contract can be written as

\[ \text{PV}[h(\tilde{c}_i)] = \frac{1}{1 + r_f} \frac{\beta^2 - a^2}{2\beta} \int_{-\infty}^{\infty} h(c) e^{-a(c - \mu) - \beta(c - \mu)} dc = \frac{1}{1 + r_f} \int_{-\infty}^{\infty} h(c) \hat{f}(c) dc . \]  

(65)

This confirms that the risk-neutral probability density is used to compute the risk-neutral expectation which is then discounted at the interest rate to determine the value.

For this problem the risk-neutral distribution is also Laplace, but it is an asymmetric Laplace. The distribution is skewed left as the right tail has a faster rate of decline. The risk-neutral mean is \( \hat{\mu} = \mu - (\beta - a)^{-1}/\ln(1 + a/\beta) \). This is smaller than the actual mean, as usual, because the risk-neutral mean does not include the risk premium. The risk-neutral variance is

\[ \hat{\sigma}^2 = 2(\beta^2 + a^2)/(\beta^2 - a^2)^2 > 2\beta^2 = \sigma^2 , \]  

which is larger than the natural variance. In the normal-exponential model, the two variances are equal. But this example shows that what changes in the risk-neutral valuation is not just the parameters, but possibly the entire distribution.

The value of a contract that pays consumption, the present value of consumption itself, is

\[ p^c_0 \equiv \text{PV}(\tilde{c}_i) = \frac{1}{1 + r_f} \frac{\beta^2 - a^2}{2\beta} \int_{-\infty}^{\infty} c \cdot e^{-a(c - \mu) - \beta(c - \mu)} dc = \frac{1}{1 + r_f} \left( \mu - 2a \right) . \]  

(66)

The price is monotone decreasing in \( a \) so \( p^c_0 \) is a sufficient statistic and \( a \) can be eliminated in the valuation in equation (65) with the risk-neutral probability density not depending explicitly on \( a \). However, the resulting valuation has no simple interpretation like those in the CARA-normal or CRRA-lognormal models. Even after \( a \) is eliminated in \( \hat{\mu} \), the actual mean \( \mu \) still appears, In the CARA-normal and CRRA-lognormal model there was no need to know either the risk aversion or the natural expected rate of return. That is a huge convenience in the valuation problem.

**The Log-Linear Approximation**

Unfortunately, current research has shown that lognormality misses important features of returns. Nevertheless, the results of the lognormal-CRRA model can be approximately true even
The theory and our intuition describe expectations. However, the results in (40) through (45) are expressed using moments of the logarithms of the relevant variables. So it is important to know how \( \mathbb{E}[\ln(1 + \tilde{r})] \) and \( \mathbb{E}[\ln(1 + \tilde{r})] \) differ. For any positive random variable, \( \tilde{z} \), this difference is the distribution’s entropy,

\[
\mathcal{H}(\tilde{z}) = \ln \mathbb{E}[\tilde{z}] - \mathbb{E}[\ln \tilde{z}].
\]

(67)

By Jensen’s inequality, entropy is always nonnegative and is strictly positive whenever \( \tilde{z} \) has any variation. It is independent of scale with \( \mathcal{H}(k \tilde{z}) = \mathcal{H}(\tilde{z}) \). The logarithmic variance has a similar property

\[
\text{var}[\ln(1 + \tilde{r})] = \text{var}[\ln k + \ln \tilde{z}] = \text{var}[\ln \tilde{z}] > 0.
\]

Like the lognormal distribution, many positive random variables are created by exponentiating an underlying random variable. In this case the second term of the entropy is just the expectation of this underlying distribution. The first term of the entropy can be determined from the log of the moment generating function, known as the cumulant generating function,

\[
\mathcal{K}_n(t) = \ln(\mathbb{E}[\exp(\theta \tilde{x})]) = \sum_{n=1}^{\infty} \frac{1}{n!} \theta^n \kappa_n(\tilde{x})
\]

(68)

where \( \kappa_n(\tilde{x}) \) is the \( n \)th cumulant of the distribution. Cumulants are a type of moment. They are related to the more familiar central moments, \( \mu_n \), by

\[
\mathbb{E}[\tilde{x}] = \kappa_1, \quad \text{var}[\tilde{x}] = \kappa_2, \quad \mu_3 = \kappa_3, \quad \mu_4 = 3\kappa_2^2 + \kappa_4, \quad \mu_5 = 10\kappa_2\kappa_3 + \kappa_5, \ldots
\]

(69)

The expression gets more complicated as \( n \) increases, but only the first four cumulants are required here.

The relation between entropy of \( \tilde{z} \) and the cumulant generating function of \( \tilde{x} \equiv \ln \tilde{z} \) is

\[
\mathcal{H}(\tilde{z}) = \ln \mathbb{E}[\tilde{z}] - \mathbb{E}[\ln \tilde{z}] = \ln \mathbb{E}[\exp(\tilde{x})] - \mathbb{E}[\tilde{x}] = \mathcal{K}_n(1) - \kappa_1(\tilde{x}) = \sum_{n=2}^{\infty} \frac{1}{n!} \kappa_n(\tilde{x}).
\]

(70)

So the lead term in the adjustment for the log-CAPM is \( \frac{1}{2} \text{var}[\ln(1 + \tilde{r})] \). For the normal distribution, all cumulants above the second are zero. This means that the entropy of a lognormal random variable \( \tilde{z} \) is just exactly this first term of the sum as already seen in (39). However, the normal distribution is the only one with a finite number of non-zero cumulants. For other distributions, entropy depends on the higher moments as well, but it can often be well approximated by half the log variance.

If consumption is a positive, but not necessarily lognormal, random variable, equation (29) gives the continuously-compounded interest rate as

\[
\ln(1 + r_f) = -\ln \delta + (p - 1) \ln c_0 - \frac{\gamma - 1}{\gamma - 2} \ln(\mathbb{E}[\tilde{c}_1]) - \ln(\mathbb{E}[\tilde{c}_1^\gamma])
\]

(71)

Applying (70) and \( \ln(\mathbb{E}[\tilde{z}]) = \mathbb{E}[\ln \tilde{z}] + \mathcal{H}(\epsilon^z) \),

\[
\ln(1 + r_f) = -\ln \delta + (p - 1) \ln c_0 - \frac{\gamma - 1}{\gamma - 2} \left[ (1 - \gamma) \mathbb{E}[\ln \tilde{c}_1] + \mathcal{H}(\tilde{c}_1^{\gamma - 1}) + \gamma \ln(\mathbb{E}[\ln \tilde{c}_1]) - \mathcal{H}(\tilde{c}_1^\gamma) \right]
\]

(72)

Using only the \( n = 2 \) terms in the sum gives the approximation

\[
\ln(1 + r_f) = -\ln \delta + (p - 1) \mathbb{E}[\ln(\tilde{c}_1 / c_0)] - \frac{\gamma - 1}{2} [p(1 - \gamma) + 2\gamma - 1] \text{var}[\ln \tilde{c}_1] + \mathcal{H}(\epsilon^z).
\]

(73)

Under lognormality the remainder is zero. Otherwise, it is
\( R \equiv -\frac{\rho^+ \gamma - 1}{1 - \gamma} \sum_{n=3}^{\infty} \frac{1}{n!} \kappa_n ((1 - \gamma) \ell n \tilde{c}_1) + \sum_{n=3}^{\infty} \frac{1}{n!} \kappa_n (-\gamma \ell n \tilde{c}_1) \)

\( = \sum_{n=3}^{\infty} \frac{1}{n!} [-(\rho + \gamma - 1)(1 - \gamma)^{-n} + (-\gamma)^n] \kappa_n (\ell n \tilde{c}_1). \)  

The final equality follows from the homogeneity of cumulants, \( \kappa_n (a \tilde{z}) = a^n \kappa_n (\tilde{z}). \) Typically \( R \) is dominated by the \( \kappa_4 \) term (or the \( \kappa_3 \) term for non-symmetric distributions). The log-linear approximation drops the remainder, \( R. \)

Asset risk premiums can also be approximated using entropy. The risk premium is the same under CRRA utility and EZ preferences (or any recursive preferences with CRRA certainty equivalents). The risk premium can be determined beginning with (13) or (24). In each case, the relation is

\[ \ell n (E[\tilde{c}_1^{-\gamma} (1 + \tilde{r})]) = \ell n (1 + \tilde{r}) + \ell n (E[\tilde{c}_1^{-\gamma}]). \]

Taking logs and using the definition of entropy

\[ E[\ell n (\tilde{c}_1 - \gamma + \ell n (\tilde{c}_1 + \ell n (1 + \tilde{r}))) = \ell n (1 + \tilde{r}) + E[\ell n \tilde{c}_1] - \gamma E[\ell n \tilde{c}_1 - \gamma + \ell n (1 + \tilde{r})] \]

The log risk premium is the difference between two entropies. Using the cumulant generating functions,

\[ E[\ell n (1 + \tilde{r})] - \ell n (1 + \tilde{r}) = \gamma E[\tilde{c}_1^{-\gamma}] - \gamma E[(\tilde{c}_1^{-\gamma} (1 + \tilde{r})) = \sum_{k=3}^{\infty} \frac{1}{n!} [\kappa_n (\tilde{c}_1^{-\gamma}) - \kappa_n (\tilde{c}_1^{-\gamma} (1 + \tilde{r})))] \]

\[ = \frac{1}{2} (\text{var}[-\gamma \ell n \tilde{c}_1] - \text{var}[-\gamma \ell n \tilde{c}_1 + \ell n (1 + \tilde{r})]) + R \]

\[ = \gamma \text{cov}[\ell n \tilde{c}_1, \ell n (1 + \tilde{r})] - \frac{1}{2} \text{var}[\ell n (1 + \tilde{r})] + R \]

where

\[ R \equiv \sum_{k=3}^{\infty} \frac{1}{n!} [\kappa_n (-\gamma \ell n \tilde{c}_1) - \kappa_n (-\gamma \ell n \tilde{c}_1 + \ell n (1 + \tilde{r}))]. \]

So the log risk premium is approximately equal to the relative risk aversion multiplied by the covariance between the log of consumption and the log return along with the standard one-half variance correction.\(^{11}\)

**Testing the Approximation**

With any approximation, an obvious concern is its accuracy. Obviously this depends on the how different the actual distribution is from the lognormal. To get an idea about the possible size of the error, we can use a gamma-subordinated normal distribution in place of the normal as the underlying distribution. A gamma-subordinated normal random variable has a normal distribution with a fixed expectation and a variance that is proportional to an independent random variable with a gamma distribution. That is, \( \tilde{x} \sim N(\mu, \sigma^2 \tilde{\xi}) \) with \( \tilde{\xi} \sim \Gamma(\alpha, \beta) \). The density and moment generating functions for the gamma distribution are

\[ f(\tilde{\xi}) = [\Gamma(\alpha)]^{-1} \tilde{\xi}^{\alpha-1} e^{-\beta \tilde{\xi}} \quad \text{and} \quad \psi_\xi(\theta) \equiv E[e^{\theta \tilde{\xi}}] = (1 - \theta/\beta)^{-\alpha} \quad \text{for} \quad \theta < \beta. \]

The moment generating function for the subordinated normal is

\[ \psi_\xi(\theta) \equiv E[e^{\theta \tilde{\xi}}] = E[E[e^{\theta \tilde{\xi}} \mid \tilde{\xi}]] = E[\exp(\theta \mu + \frac{1}{2} \sigma^2 \theta^2 \tilde{\xi})] = e^{\theta \mu} \psi_\xi (\frac{1}{2} \sigma^2 \theta^2) = e^{\theta \mu} (1 - \sigma^2 \theta^2 / 2\beta)^{-\alpha}. \]

\(^{11}\) More commonly, the covariance used in the risk premium is \( \text{cov}[\ell n (\tilde{c}_1/c_0), \ell n (1 + \tilde{r})]. \) There is no difference because \( \ell n (\tilde{c}_1/c_0) = \ell n (\tilde{c}_1) - \ell n (c_0), \) and \( c_0 \) is known so the two covariances are equal.
The cumulant generating function for \( x \) is
\[
K_x(\theta) = \ell n \psi_x(\theta) = \theta \mu - \alpha \ell n(1 - \sigma^2 \theta^2 / 2\beta) 
\approx \theta \mu + \frac{1}{2} \alpha \sigma^2 \beta^{-1} \theta^2 + \frac{1}{4} \alpha \sigma^4 \beta^{-2} \theta^4 + \cdots
\] (79)

Using (69), the variance, the fourth central moment, and excess of \( \tilde{x} \) are
\[
\text{var}[\tilde{x}] = \kappa_2 = \frac{\alpha \sigma^2}{\beta}, \quad \mu_4 \equiv \mathbb{E}[(\tilde{x} - \bar{x})^4] = 3 \kappa_2^2 + \kappa_4 = \frac{3 \alpha^2 \sigma^4}{\beta^2} + 6 \frac{\alpha \sigma^4}{\beta^2} = 3 \frac{\alpha \sigma^4}{\beta^2}(\alpha + 2)
\]
\[
\text{excess} = \frac{\mu_4}{\mu_2^2} - 3 = 3 \left( \frac{\alpha + 2}{\alpha} \right) - 3 = \frac{6}{\alpha}.
\]
(80)

So the smaller is \( \alpha \) the more leptokurtotic or fat-tailed is \( \tilde{x} \). As \( \alpha \to \infty \), the excess goes to zero and the distribution approaches the normal.

Suppose \( \ell n \tilde{c}_i \) has a gamma-subordinated normal distribution described above for \( \tilde{x} \), then from (79) and (80),
\[
\mathbb{E}[\ell n \tilde{c}_i] = \mu \quad \text{and} \quad \text{var}[\ell n \tilde{c}_i] = \alpha \sigma^2 / \beta.
\]
The interest rate is determined using (29).
\[
1 + r_f = \frac{c_0^{\rho^{-1}}}{\delta(\mathbb{E}[\tilde{c}_1^{-1 - \gamma}]^{(\rho+\gamma-1)/(\rho-1)} \times \mathbb{E}[\tilde{c}_1^{-\gamma}])} = \frac{c_0^{\rho^{-1}}}{\delta(\psi_{\mu_{\tilde{c}_1}}(1 - \gamma))^{(\rho+\gamma-1)/(\rho-1)} \times \psi_{\mu_{\tilde{c}_1}}(-\gamma)} \]
\[
= \frac{c_0^{\rho^{-1}}}{\delta e^{(\rho+\gamma-1)\mathbb{E}[\ell n \tilde{c}_1]_1}(1 - (1 - \gamma) \var{\ell n \tilde{c}_1}_2/2\alpha)^{-\alpha(\rho+\gamma-1)/(\rho-1)} e^{-\gamma \mathbb{E}[\ell n \tilde{c}_1]_2}(1 - \gamma^2 \var{\ell n \tilde{c}_1}_2/2\alpha)^{-\alpha}}.
\]
(81)

The continuously compounded rate is
\[
r_{\text{cont-comp}} = -\ell n \delta + (1 - \rho)(\mathbb{E}[\ell n \tilde{c}_1] - \ell n c_0) + \frac{\alpha(\rho+\gamma-1)}{1 - \gamma} \ell n \left(1 - \frac{\alpha}{2} (1 - \gamma) \var{\ell n \tilde{c}_1}_2/2\alpha\right) + \alpha \ell n \left(1 - \frac{1}{2} \gamma^2 \alpha^{-1} \var{\ell n \tilde{c}_1}_2/2\alpha\right).
\]
(82)

To determine the exact log risk premium under the gamma-subordinated normal distribution assume that \( \ell n \tilde{c}_i \) and \( \ell n (1 + \tilde{r}) \) are normally distributed with the same gamma subordination, \( \xi \). Define the random variable \( \tilde{z} = -\gamma \ell n \tilde{c}_i + \ell n (1 + \tilde{r}) \). It has a normal distribution conditional on the subordinating variable \( \xi \). Let \( \omega^2 \) denote the variances of the underlying normals. Then the conditional and unconditional variances and covariances are
\[
\text{var}[\ell n \tilde{c}_i | \xi] = \omega^2 \xi, \quad \text{var}[\ell n (1 + \tilde{r}) | \xi] = \omega^2 \xi, \quad \text{cov}[\ell n \tilde{c}_i, \ell n (1 + \tilde{r}) | \xi] = \omega \omega_c, \xi
\]
\[
\text{var}[\ell n \tilde{c}_i] = \alpha \omega^2 / \beta, \quad \text{var}[\ell n (1 + \tilde{r})] = \alpha \omega^2 / \beta, \quad \text{cov}[\ell n \tilde{c}_i, \ell n (1 + \tilde{r})] = \alpha \omega_c / \beta.
\]
(83)

From (24) the exact pricing relation for the risky asset is
\[
\bar{\lambda} = \mathbb{E}[^{\tilde{c}_i^{-\gamma}}(1 + \tilde{r})] = \mathbb{E}[e^{\tilde{z}}] = e^\tilde{z}(1 - \omega^2 / 2\beta)^{\alpha}
\]
\[
= \exp(-\gamma \mathbb{E}[\ell n \tilde{c}_1] + \mathbb{E}[^{\ell n (1 + \tilde{r})}][1 - (\gamma^2 \omega^2_c - 2\gamma \omega_c + \omega^2_c)^{2}\beta]^{-\alpha}.
\]
(84)

For the risk-free asset, the exact relation is
\[
\bar{\lambda} = \exp(-\gamma \mathbb{E}[\ell n \tilde{c}_1] + \ell n (1 + \tilde{r}))(1 - \gamma^2 \omega^2_c / 2\beta)^{-\alpha}.
\]
(85)

Subtracting (85) from (84) gives the exact log risk premium
\[
\mathbb{E}[\ell n (1 + \tilde{r})] - \ell n (1 + r_f) = \alpha \ell n \left(1 - (\gamma^2 \omega^2_c - 2\gamma \omega_c + \omega^2_c)^{2}\beta\right) + \alpha \ell n \left(1 - \gamma^2 \omega^2_c / 2\beta\right)
\]
\[
= \alpha \ell n \left\{1 - (\gamma^2 \var{\ell n \tilde{c}_1}_2 - 2\gamma \text{cov}[\ell n \tilde{c}_1, \ell n (1 + \tilde{r})] + \var{\ell n (1 + \tilde{r})}_2) / 2\alpha\right\}
\]
\[
- \alpha \ell n \left(1 - \gamma^2 \var{\ell n \tilde{c}_1}_2 / 2\alpha\right).
\]
(86)
The figure shows the relation between the excess kurtosis and the approximation errors in the interest rate and the risk premium. The parameters used are $\rho = 1$, $\gamma = 5$, $\text{var}[\ln \tilde{c}_1] = 0.001$, $\text{var}[\ln(1+\tilde{r}_1)] = 0.25$, and $\text{corr}[\ln \tilde{c}_1, \ln(1+\tilde{r}_1)] = 0.1$. The risk-premium error is independent of $\rho$, and the interest rate error is independent of $\gamma$ and $\text{var}[\ln(1+\tilde{r}_1)]$. Otherwise, both errors are decreasing in the correlation and $\rho$ and are increasing in $\gamma$, $\text{var}[\ln \tilde{c}_1]$, and $\text{var}[\ln(1+\tilde{r}_1)]$. So the errors show in the figure should be higher than the actual errors in typical cases. The interest rate errors are certainly negligible, never amounting to more than a half a basis point. The risk premium errors can be economically meaningful if the excess kurtosis is large.

Backward Induction (Stochastic Dynamic Programming) Solution

All of the pricing results have been derived here without solving the consumption-portfolio problem. This is fortunate as, in general, the solution of these problems usually requires numerical procedures. The general method is described below and the next section solves a simple problem analytically.

The solution method is a two-step procedure. Time-0 consumption is “fixed” leaving only the choice of the optimal portfolio. Then the expected utility of time-1 consumption, $\tilde{c}_1 = (W_0 - c_0)\tilde{R}(c_0)$, is maximized by choosing the optimal portfolio, $w$, with gross return, \( \tilde{R}(c_0) \equiv 1 + w'(c_0)\tilde{r} \).

$$\max_w \mathbb{E}[V_1(\tilde{R}(c_0))] = \mathbb{E}[U(c_0, (W_0 - c_0)(1 + w'(c_0)\tilde{r})]).$$ (87)

In general, the solution will depend on the amount of time-0 consumption. Once the optimal portfolio is known for all $c_0$ choices, $\tilde{R}^*(c_0)$, the optimal time-0 consumption can be determined by maximizing

$$\max_{c_0} \mathbb{E}[U(c_0, (W_0 - c_0)\tilde{R}^*(c_0))].$$ (88)

---

12 Dew-Becker (2015) reports the variance of consumption growth from various studies that range from 0.00016 (Campbell and Cochrane, 1999) to 0.0081 (Colacito and Croce, 2011). $\text{Var}[\ln(1+\tilde{r}_1)] = 0.25$ corresponds to a return standard deviation of 50%. 

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Figure 7.1: Errors in the Log-Linear Approximation

This figure shows the errors in the interest rate and the risk premium as a function of the excess kurtosis.
The first-order condition is
\[
\frac{\partial \mathbb{E}[V_0(c_0)]}{\partial c_0} = \mathbb{E} \left[ U_1(\cdot) - U_2(\cdot)(\tilde{R}^*(c_0) - (W_0 - c_0)\frac{\partial \tilde{R}^*(c_0)}{\partial c_0}) \right].
\] (89)

For some utility functions, the problem is greatly simplified. For example with time-additive CRRA utility, the portfolio problem in (87) is to maximize \(\delta(W_0 - c_0)^{1-\gamma} \mathbb{E}[\tilde{R}^{1-\gamma}]/(1 - \gamma)\). The optimal portfolio is independent of the multiplier \(\delta(W_0 - c_0)^{1-\gamma}\) so it does not depend on the amount consumed at time 0 nor on the initial wealth. This means that only one portfolio problem needs to be solved rather than one for every possible amount of time-0 consumption. Optimal time-0 consumption then maximizes \((c_0^{1-\gamma} + \delta(W_0 - c_0)^{1-\gamma} \mathbb{E}[\tilde{R}^{1-\gamma}])/(1 - \gamma)\) with \(\tilde{R}\) having a fixed distribution the same for all \(c_0\). All LRT utility functions simplify the dynamic program in a similar fashion as seen in the example below and discussed in complete detail in Chapter ??.

The optimal portfolio and consumption for recursive preferences utility can be determined in the same fashion. Recursive preferences consist of a utility function, \(v\), that describes risk preferences over time-1 consumption and an aggregator function, \(\Gamma\), describing substitution across time.

\[
\psi_0 = \Gamma(c_0, \psi_1) \quad \text{with} \quad \psi_1 = \Psi(\tilde{c}_1) \equiv v^{-1}(\mathbb{E}[v(\tilde{c}_1)]),
\] (90)

which is the certainty equivalent of time-1 consumption. As in (87) the utility of time-1 consumption can be converted to a utility function defined over portfolio returns \(v((W_0 - c_0)\tilde{R})\).

Optimal time-0 consumption is determined as in (88), \(V(c_0) \equiv \Gamma(c_0, \Psi((W_0 - c_0)\tilde{R}))\).

This backward induction procedure is also used in multi-period and even infinite horizon models. It is also useful in the valuation of derivative contracts for which decisions are involved.

**A Sample Portfolio Problem with LRT Utility and EZ Preferences**

Complete markets and mean-variance portfolio problems can often be solved completely. However, for this more general model only numerical solutions are usually possible. Very simple models, like the one below, can be solved analytically.

In this model, there are two assets, a riskless asset with gross return per dollar return of \(R_f \equiv 1 + r_f\) and a risky asset whose gross return has three outcomes, \(H\), \(L\), and \(R_f\) with probabilities \(\pi_H\), \(\pi_L\), and \(1 - \pi_H - \pi_L\), respectively. To avoid arbitrage, \(L < R_f < H\). As there are three states and only two assets, the market is incomplete.

Preferences are modified EZ preferences. The same constant elasticity aggregator is used, but the utility evaluation is LRT for time-0 and time-1 consumption, \(u(c_t) = (c_t - \tilde{c}_t)^{-\gamma}/(1 - \gamma)\) with a subjective discount factor of \(\delta\) for time-1 utility.

\[
(W_0 - c_0)^{1-\gamma}(1 + r_f)^{\gamma} - \tilde{c}_1/(1 - \gamma).
\] From (20), the first-order condition for the fraction, \(w\), of money invested in the risky asset is

\[
0 = \pi_H \left[ (W_0 - c_0)[wH + (1 - w)R_f - \tilde{c}_1]^{1-\gamma} (H - R_f) \right] - \pi_L \left[ (W_0 - c_0)[wL + (1 - w)R_f - \tilde{c}_1]^{1-\gamma} (L - R) \right].
\] (91)

So the optimal portfolio holds

\[
\frac{w^*(\gamma, \tilde{c}_1)}{R_f(W_0 - c_0)} = \left[ 1 - \frac{\tilde{c}_1}{R_f(W_0 - c_0)} \right] \times \frac{\pi_L (R - L)\tilde{c}_1^{1-\gamma} - \pi_H (H - R_f)\tilde{c}_1^{1-\gamma}}{\pi_H (H - R_f)^{1-\gamma} - \pi_L (R_f - L)^{1-\gamma}}. \] (92)

There are two factors. Only the first depends on time-0 consumption and wealth, and it is equal to 1 when utility is CRRA with \(\tilde{c}_1 = 0\). So the optimal portfolio of a CRRA investor with a given risk aversion is
\[ w^*(\gamma, 0) = R_f \left[ \frac{\pi_L (R_f - L)}{\pi_H (H - R_f)} \right]^{1/\gamma} - \left[ \frac{\pi_H (H - R_f)}{\pi_H (H - R_f)} \right]^{1/\gamma} \]

independent of wealth and time-0 consumption. The dollar holding of the risky asset scales in proportion to the amount invested. This means that the portfolio problem need not be solved for every possible amount invested. This simplification is one of the reasons that CRRA utility is so commonly used in models. In the next chapter this property is shown to hold in general for CRRA investors regardless of the investment opportunities.

The portfolio of an LRT investor with \( \xi_1 \neq 0 \) does depend on the amount invested, but it can be separated into two parts. First, \( \xi_1/R_f \) dollars are invested in the safe asset. This is the fraction \( \xi_1/[R_f(W_0 - c_0)] \) of the entire investment. This purchase assures that the investor’s time-1 wealth, and therefore his time-1 consumption, is at least \( \xi_1 \), provided the remainder of the portfolio has limited liability. This is important because marginal utility becomes infinite as \( \xi_1 \) falls to \( \xi_1 \). The remaining wealth is invested with the fraction \( w^*(\gamma, 0) \) in the risky asset and \( 1 - w^*(\gamma, 0) \) in the safe asset. So again the portfolio problem is simplified for LRT investors.

Because the exponents in the numerator of \( w^* \) are negative, the holding in the risky asset is positive if and only if \( \pi_L (H - R_f) > \pi_H (R_f - L) \); that is, \( w^* > 0 \) whenever \( \mathbb{E}[\tilde{r}] > r_f \). This is essentially the same result stated in Chapter 2 in the section on first-order risk aversion. There it was shown that some fraction of a risky prospect is always accepted provided it has a positive expected payoff. Here the expected rate of return must exceed the interest rate rather than zero because the alternative to not investing at risk is earning the interest rate. The investor takes a short position in the risky asset when the expected rate of return is less than \( r_f \). An investment of exactly zero is the right choice only when the expected rate of return is equal to the interest rate. If the expected rate of return is sufficiently high, the investor might buy the risky asset on margin. The CRRA portfolio has limited liability for any \( w < R_f/(R_f - L) \) so some margin buying is possible whenever \( L > 0 \). When \( \xi_1 > 0 \), \( L \) must have a higher value to permit buying on margin.

Assuming the expected excess rate of return is positive, the fraction of wealth invested at risk is increasing in \( L \) and \( \pi_H \) and decreasing in \( \pi_L \). All of these changes are first-order stochastic dominance improving. What is surprising, perhaps, is that an increase in \( H \), which is also first-order stochastic dominance improving, does not always increase \( w^* \). The figure plots the optimal portfolio weight, \( w^* \), as a function of the better outcome, \( H \), for \( \pi_H = \pi_L = 0.25 \), \( L = 0.8 \), \( R_f = 1 \), and \( \xi_1 = 0 \). The portfolio weight is strictly increasing for \( \gamma = 0.5 \); however, for \( \gamma = 2 \), \( w^* \) increases to a maximum of 0.858 at \( H = 2.166 \) and then decreases. Why? The answer to this question is that CRRA utility is a bounded negative function when \( \gamma > 1 \), so when \( H \) is large, it contributes approximately 0 to utility. Increasing \( H \) further makes little change to expected utility. The investor, therefore, chooses to “spend” this bonus to increase his payout in the poor state. He does this by shifting his portfolio towards the safe asset. Although this decreases the portfolio’s expected rate of return, it is beneficial because it increases the utility realized in the worst state, when marginal utility is highest.

With the optimal portfolio problem solved, time-0 consumption can be determined. The exact description of the assets is no longer important provided the return on the optimal portfolio is known. Define \( \tilde{R}_f \equiv 1 + w^*(\gamma, 0) \tilde{r} \) to be the gross return on the optimal portfolio for an investor with a CRRA of \( \gamma \). An investor with LRT utility buys \( \xi_1 / R_f \) dollars of the risk-free asset and

\(^{13}\)\text{If } (W_0 - c_0)R_f < \xi_1, \text{ no portfolio can be constructed guaranteeing that } \xi_1 \geq \xi_1 \text{ so expected utility is undefined for every possible portfolio, and the problem has no solution.}

\(^{14}\)The comparative static is \( \partial w^*/\partial H \propto [\pi_H (H - R_f)]^{-2/\gamma} + [\pi_H (H - R_f)]^{1/\gamma}[(R - L)(H - R_f)/(H - R_f)]\gamma - (1-1/\gamma)] \]. All terms and factors are positive except possibly the last, \(-1(1-1/\gamma)\). Therefore, a necessary condition in this model for \( w^* \) to decrease with an increase in \( H \) is that relative risk aversion exceeds one \((\gamma > 1)\).
invests $W_0 - c_0 - c_1/R_f$ in the portfolio $w(\gamma, 0)$. Wealth available at time 1 and therefore consumption is $\tilde{W}_1 = (W_0 - c_0 - c_1/R_f)\tilde{R}_\gamma + c_1$. Realized utility is $(\tilde{W}_1 - c_1)^{-\gamma} = [(W_0 - c_0 - c_1/R_f)\tilde{R}_\gamma]^{-\gamma}$.

The $c_0$ maximand for recursive utility is
\[
\psi_0 = (c_0 - c_0)^p + \delta \left( E[(\tilde{W}_1 - c_1)^{-\gamma}] \right)^{p/\gamma} = (c_0 - c_0)^p + \delta (W_0 - c_0 - c_1/R_f)^p \left( E[\tilde{R}_\gamma] \right)^{p/\gamma}.
\]

Time-additive utility is included as the special case $\rho = 1 - \gamma$. The first-order condition and solution are
\[
0 = \rho(c_0 - c_0)^{p-1} - \delta \rho(w_0 - c_0 - c_1/R_f)^{p-1} E_\gamma^p
\]
\[
\Rightarrow c_0^* = \frac{c_0 + (\delta E_\gamma^p)^{1/(p-1)}(W_0 - c_1/R_f)}{1 + (\delta E_\gamma^p)^{1/(p-1)}}
\]
where $E_\gamma^* \equiv (E[\tilde{R}_\gamma])^{1/(1-\gamma)}$. This is the certainty equivalent gross return for a CRRA investor with relative risk aversion of $\gamma$. He would be indifferent between choosing a portfolio with random gross return $\tilde{R}_\gamma$ and a certain gross return of $E_\gamma^*$. 7

If $\rho = 0$, then the maximand is $\psi_0 = (c_0 - c_0)^p + \delta \ln(W_0 - c_0 - c_1/R_f) + \delta \ln E_\gamma$ after application of L’Hospital’s rule. Optimal consumption is $c_0^* = (1 + \delta)^{-1}(W_0 + \delta C_0 - c_1/R_f)$, which depends on neither the investment opportunities nor the risk aversion. This same result was seen in complete market and is quite general. For recursive preferences with unitary ESIC, consumption is independent of the investment opportunities regardless of the risk aversion.

The marginal propensity to consume out of time-0 wealth, $(\text{MPC})$, is constant for EZ aggregation preferences
\[
(MPC) \equiv \frac{\partial c_0}{\partial w_0} = \frac{(\delta E_\gamma^*)^{1/(p-1)}}{1 + (\delta E_\gamma^p)^{1/(p-1)}}.
\]
This is also the average propensity to consume if $c_0 = c_1 = 0$. The comparative statics for the propensity to consume are
\[
\frac{\partial (\text{MPC})}{\partial \delta} = -\frac{\delta (2-p)(p-1)}{(1-\rho)[1 + \delta (p-1)E_\gamma^p(p-1)]^2} < 0
\]
\[
\frac{\partial (\text{MPC})}{\partial E_\gamma^*} = -\rho \frac{(\delta E_\gamma^*)^{1/(p-1)}}{(1-\rho)[1 + (\delta E_\gamma^p)^{1/(p-1)}]^2} \geq 0 \quad \text{as} \quad \rho \leq 0
\]
\[
\frac{\partial (MPC)}{\partial \rho} = -\frac{\delta^{1/(\rho-1)} E^p (\rho-1)^{-1} f_n (\delta E^*)}{[1 + \delta^{1/(\rho-1)} E^p (\rho-1)]^{2/\rho} (1 - \rho)^2} \quad \geq 0 \quad \text{as} \quad \delta E^* \leq 1. \tag{99}
\]

The result in (97) is obvious. Time preference is measured by \( \delta \). An increase in \( \delta \) increases the utility contribution of time-1 consumption relative to that of time-0 consumption. This induces the investor to save more and consume less.

The result in (98) illustrates the standard income and substitution effects of price theory. \( E^* \) is a probabilistic measure of how much time-1 consumption is available per unit of time-0 consumption. In that sense \( 1/E^* \) is the “price” of time-1 consumption relative to time-0 consumption. Increasing \( E^* \) lowers the price of time-1 consumption causing a substitution out of time-0 consumption. But lowering the price also increases the effective wealth so there is an “income” effect increasing consumptions at both times. The large is \( \rho \) the higher is the elasticity of consumption, \( (1 - \rho)^{-1} \), and the greater is the substitution effect. When \( \rho > 0 \), the substitution effect dominates, and time-0 consumption falls when \( E^* \) increases.

The final result in (99) illustrates the effects of an increase in elasticity. The more elastic is the investor’s demand (the higher is \( \rho \)), the more is consumption shifted to whichever option gives the most utility per dollar spent. The relative benefit of time-1 consumption is measured by \( \delta E^* \) is the reciprocal of the cost of time-1 consumption. So when \( \delta E^* > 1 \), time-1 consumption is cheaper that time-0 consumption, and an increase in \( \rho \) shifts consumption to time 1.

**The Fundamental Theorem of Asset Pricing — Once Again**

With the portfolio problem solved, at least in theory, the Fundamental Theorem of Asset Pricing in Chapter 3 can be extended to add another equivalent condition. In addition, an important corollary can be proved.

**Theorem 7.1: Fundamental Theorem of Asset Pricing.** The following three statements are equivalent:

(i) There are no arbitrage opportunities: \( (\exists \eta) : \Xi \eta > 0 \)

(ii) There exists a strictly positive linear pricing rule: \( (\exists q \gg 0) : p = Xq \).

(iii) There exists an agent with strictly increasing, concave, time-additive utility who has an optimal solution to the portfolio problem.

**Proof:** Statements (i) and (ii) were shown to be equivalent in Theorem 3.3. That (iii) \( \Rightarrow \) (i) is immediate. If there were any arbitrage opportunities, consumption could be increased in some states or at time 0 without decreasing any other consumption so there could not be an optimum. Therefore, the existence of an equilibrium means there must be no arbitrage. That (ii) \( \Rightarrow \) (iii) can be shown by construction. Equation (4) is a standard constrained convex optimization problem. It has the solution provided the first order conditions in (5) can be satisfied. Choose the utility function \( U(c_0, c_1, s) \equiv -\exp(-a_0 c_0) - \exp(-a_1 c_1) \). Because both first derivatives drop from \( \infty \) to 0 over the real line, the first-order conditions must be satisfied somewhere.\(^{15}\)

The reason for picking the exponential function in the proof is one of convenience. No assumption has been made that assets have limited liability. When negative payoffs are possible on every asset, it might be impossible to form a portfolio with a strictly nonnegative payoff. Obviously there could not be any arbitrage, but also many utility functions like log would have

\(^{15}\) With a continuous outcome space, the probability space needs to be complete to ensure an optimum. The notion of a complete probability space is beyond the scope of this book; however, all of the probability measures commonly used in Finance are complete.
undefined expectations for all portfolios so there could not be an optimal portfolio. To cover the
general case, utility must be defined over negative consumption; the exponential function is
merely a convenient concave utility function defined over the entire real line. If additional
restrictions are made about the payoff matrix, like all assets have limited liability and there is no
state in which all assets have payoffs of zero, then log utility and other CRRA utilities will also
have optimal solutions.

The third statement in the theorem above might appear to be a trivial addition to the
linear pricing rule. Its importance is that whenever a set of payoffs and prices is free from
arbitrage, there must be some equilibrium that supports those prices as shown in the next
theorem.

**Theorem 7.2: The Absence of Arbitrage Guarantees an Equilibrium.** Any set of
payoffs, $X$, and prices, $p$, that are free of arbitrage can be supported in some equilibrium for any
set of state probabilities, $\pi \gg 0$.

**Proof:** As only existence is required, we are free to choose the characteristics of the
investors. Assume there are $K$ investors with homogeneous beliefs and the same utility function
$U(c_0, c_t, s) = -(1/a_0) \exp(-a_0 c_0) - (1/a_s) \exp(-a_s c_s)$. Let $\overline{\eta}$ denote the per capita supply of the
assets. Then the per capita consumption available in the states at time one is $c = X \overline{\eta}$. Assume for
the moment that the market is complete and the state prices are $q_s = \pi_s \exp(a_0 \overline{c}_0 - a_s \overline{c}_s)$. Then it is
optimal for each investor to consume per capita consumption in each state. This allocation can be
achieved by each investor holding the same portfolio, $\overline{\eta}$, so the availability constraint is not bind-
ing. Furthermore, the complete market constraint, the no-arbitrage condition, and the definition
of per capita consumption guarantee that the incomplete market budget constraint is also
satisfied

$$W_0 = \overline{c}_0 + q' \overline{c} = \overline{c}_0 + q' X \overline{\eta} = \overline{c}_0 + p' \overline{\eta}.$$  \hspace{1cm} (100)

Because $\overline{\eta}$ is the optimal portfolio in a complete market without the availability constraint and it
is feasible in any market, it must be optimal in the constrained market. This proves the existence
of an equilibrium.

The importance of this theorem is that it demonstrates that arbitrage and equilibrium are
somewhat disconnected. There cannot be an equilibrium if there is arbitrage, but the absence of
arbitrage puts only very minor restrictions on the possible equilibria. In particular, without
further information, it is impossible to deduce the state probabilities, and therefore expected rates
of return, from the absence of arbitrage alone. We will see one example of the type of additional
information which makes this recovery possible in Chapter 16.