Chapter 4 — Complete Markets

The Basic Complete-Market Problem

There are $N$ primary assets each with price $p_i$ and payoff $X_{si}$ is state $s$. There are also a complete set of $S$ pure state securities each of which pays $1$ in its own state $s$ and $0$ otherwise. These pure state securities are often called Arrow-Debreu securities. Assuming there is no arbitrage, the price of the pure state $s$ security must be $q_s$, the state price. In a complete market like this, investors will be indifferent between choosing a portfolio from the $S$ state securities and all $N + S$ securities.

A portfolio holding $n$ shares of primary assets and $\eta_1$ shares of the Arrow-Debreu securities will provide consumption $c = Xn + \eta_1$ in the states at time 1. A portfolio holding $\eta_2 = Xn + \eta_1$ of just the Arrow-Debreu securities obviously gives the same consumption. The costs of the two portfolios are $q'c$ and $p'n + q\eta_1$. From the definition of $c$ and the no-arbitrage result that $p' = q'X$, these are obviously the same.

Of course in equilibrium the original primary assets must actually be held. However this can be done by financial intermediaries like mutual funds. They buy all the primary shares financing the purchases by issuing the Arrow-Debreu securities. By issuing $h = Xn$ Arrow-Debreu securities where $n$ is the aggregate supply of the primary assets, the intermediary is exactly hedged, and their role can be ignored.

So with no loss of generality, we can therefore assume that individual investors only purchases Arrow-Debreu securities ignoring the primary assets. This permits picking consumption in each state, $c = \eta_1$, directly ignoring the availability constraint. The budget-only-constrained portfolio problem for an investor is

$$\max_{c_0, c_1} \sum_s \pi_s (c_0, c_1, s) + \lambda(W_0 - c_0 - c'q).$$

(1)

This is a completely general individual specification for Expected Utility Theory. Utility depends on current consumption, $c_0$, random future consumption, and possibly explicitly on the state realized. The probabilities used are the investor’s subjective beliefs which may or may not be correct or agreed upon by different investors. EUT imposes the structure. Utility must be linear in the probabilities, and the utility realized in each in each state can depend only on consumption in that state and at time 0. The consumption that would have ensued had another state occurred does not affect the realization of utility. Also, although the presentation appears to be that of a single-period model, it is easily interpreted as a multi-period model as well.\(^1\)

If utility is differentiable, the first-order conditions are

$$0 = \frac{\partial \mathcal{L}}{\partial c_i} = \pi_i \left. \frac{\partial U(c_0, c_1, s)}{\partial c_i} \right|_{c_i = c_i} - \lambda q_s = \pi_i U_1(c_0, c_1, s) - \lambda q_s$$

$$0 = \frac{\partial \mathcal{L}}{\partial c_0} = \mathbb{E} \left[ \left. \frac{\partial U(c_0, c_1, s)}{\partial c_0} \right|_{c_0 = c_0} \right] = \mathbb{E} \left[ U_0(c_0, c_1, s) \right] - \lambda.$$  

(2)

where the subscripts, 1 and 0 denote differentiation with respect to $c_1$ and $c_0$, respectively. Combining the first-order conditions

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\(^1\) Expected utility can be expressed as $\sum_h \pi_h U(c_0, c_3, h)$ where $h$ is a history of the set of states and $c_3$ is a vector of length $T$ giving consumption at each date. If utility is time-additive, then expected utility can be expressed more easily as $\sum_t \sum_{h(t)} \pi(h(t)) U(c(t), t, s(t))$. This is the same as (1) except that the total of all the probabilities is equal to $T$. 

These results are reminiscent of standard price theory where the marginal rate of substitution is equal to the price ratio. The only difference here is that expectations are involved because the state is uncertain.

The stochastic discount factor, $\tilde{m}$, is the random variable that has realizations equal to the state-price per unit probability is. So the stochastic discount factor is proportional to the marginal utility of consumption, $\tilde{m} = \lambda^{-1} U_1(c_0^*, \tilde{c}_1^*, \tilde{s})$. The Lagrange multiplier, $\lambda$, is the cost of the budget constraint so it is positive. Marginal utility is assumed to be decreasing so the inverse function is decreasing as well. That means that when the utility function, and therefore the inverse marginal utility function, is state-independent, optimal consumption is decreasing in the state-price per unit probability whatever the time-0 consumption is. Furthermore, if two investors have the same beliefs, then their consumption aligns perfectly across states because both are inversely aligned with $q_s/\pi_s$. This important result is formalized in Theorem 4.1 below.

**Theorem 4.1: Co-monotonicity of Consumption in a Complete Market.** In a complete market, every risk averse investor with state-independent utility and homogeneous beliefs has time-1 consumption that is monotonically decreasing in the SDF and is, therefore, monotonically aligned with the time-1 consumption of every other such investor. In addition, every pattern of consumption that is monotonically decreasing in the SDF is the optimal portfolio for some risk averse, state-independent, time-additive utility function.

**Proof:** The first portion of the proof follows directly from (2) and the decreasing marginal utility. From the first-order condition, the marginal utility of time-1 consumption in state $s$ is proportional to $q_s/\pi_s$, so for any two investors, $k$ and $\kappa$, the ratio of marginal is the same in two states, $s$ and $s'$

$$
\frac{\partial U_k(c_{0k}, c_{sk})}{\partial c_1} / \frac{\partial U_\kappa(c_{0\kappa}, c_{s\kappa})}{\partial c_1} = \frac{q_{s}}{\pi_{s}} = \frac{q_{s'}}{\pi_{s'}}
$$

When marginal utility is strictly decreasing this guarantees

$$
c_{sk} > c_{\kappa k} \Rightarrow \frac{\partial U_k(c_{0k}, c_{sk})}{\partial c_1} < \frac{\partial U_k(c_{0k}, c_{s\kappa})}{\partial c_1} \Rightarrow \frac{q_{s}}{\pi_{s}} < 1
$$

Therefore, in a complete market of investors with homogeneous beliefs and state-independent utility, time-1 consumption aligns perfectly across states being strictly decreasing in the state price per unit probability, $m_s = q_s/\pi_s$.

To prove the second part let $c$ be the $S$-vector of optimal consumption for some investor with utility function $U(c_0, c_s)$ and let $z$ be any other $S$-vector whose elements are in the same order. Define the function $v'(z) \equiv U_1(c_0, c_1)$. By construction $v'(z) \equiv U_1(c_0, c_1) = \lambda q_s/\pi_s$ from (2), and it is positive for all $z_1$ as $\lambda, q_s$, and $\pi_s$ are. Furthermore as $c$ and $z$ align perfectly $v'$ is decreasing in $z$ because $U_1$ is decreasing in $c$. Therefore, $v$ is concave and a valid risk averse marginal utility function.

For time-additive, state-independent utility, $u(c_0) + \delta u(c_1)$, the first-order conditions are

$$
0 = \pi_s \delta u'(c_1) - \lambda q_s \quad 0 = u'(c_0) - \lambda
$$
with optimal solution
\[ c_s^* = u^{-1}(\lambda q_s / \pi_s) \quad c_0^* = u^{-1}(\lambda). \] (7)

All consumption is given explicitly in terms of \( \lambda \). However, it typically requires solving a non-linear equation to determine \( \lambda \).

\[ w_0 = c_0^*(\lambda) + \sum_s q_s c_s^*(\lambda). \] (8)

**Portfolios for CRRA and EZ Preferences**

For commonly used utility functions, (7) and (8) can usually be solved analytically. For example, for CRRA utility with \( u'(c) = c^{-\gamma} \) and \( u'^{-1}(y) = y^{1/\gamma} \), state \( s \) consumption is

\[ c_s = (\lambda q_s / \pi_s)^{-1/\gamma} = (c_0^{-1/\gamma} q_s / \pi_s)^{-1/\gamma} = c_0(\delta \pi_s / q_s)^{1/\gamma}. \] (9)

The second equality substitutes \( \lambda = u'(c_0) = c_0^{-1/\gamma} \). Using the budget constraint (8), completes the solution by determining \( c_0 \).

\[ W_0 = c_0 + \sum_s q_s c_s = c_0 + c_0 \delta^{1/\gamma} \sum_s q_s^{1/\gamma} \pi_s^{1/\gamma} \] \[ \Rightarrow c_0 = \left(1 + \delta^{1/\gamma} \sum_s q_s^{1/\gamma} \pi_s^{1/\gamma}\right)^{-1} W_0. \] (10)

From (9), the consumption in every state is proportional to time-0 consumption, and from (10) consumption at time 0 is also proportional to wealth. Therefore, as the investor’s initial wealth increases, the consumption in every state and at time 0 increases in proportion. This is an important property of CRRA utility and explains why it is used in many models. An economy can be analyzed in a steady state where grown does not affect trade-offs.

As previously noted, consumption is smaller in states where the SDF has a higher realization so consumption increases with the state probability and decreases with the state price. The expenditure made to create state-\( s \) consumption is \( \xi_s = q_s c_s = c_0(\delta \pi_s)^{1/\gamma} q_s^{1/\gamma}. \) This is also increasing in the state probability, but it is decreasing in the state price when \( \gamma \) is less than 1.

For state-independent Epstein-Zin preferences, \( \Psi_0 \equiv \rho^{-1/\gamma} \sum_s \pi_s c_s^{1/\gamma} (1 - \gamma) \) for the maximization problem is \( \max \ L \equiv \rho^{-1/\gamma} \left[ c_0^0 + \delta(\sum_s \pi_s c_s^{1/\gamma}) (1 - \gamma) \right] + \lambda(W_0 - c_0 - c'q). \) (11)

The first-order conditions are

\[ 0 = \frac{\partial L}{\partial c_0} = c_0^{\rho-1} - \lambda \quad 0 = \frac{\partial L}{\partial c_s} = \delta(\sum_s \pi_s c_s^{1/\gamma}) (1 - \gamma) - \lambda q_s \] (12)

From the time-1 conditions, we have that state-\( s \) consumption is \( c_s = K(q_s / \pi_s)^{-1/\gamma} \); this is proportional to \( (q_s / \pi_s)^{-1/\gamma} \) just as with CRRA utility. This means that the relative consumption in any two states is the same for CRRA and EZ investors who have the same risk aversion. The portfolios of these investors hold the assets in the same proportion as CRRA investors, but they differ in the time-0 consumption and the amount invested when the initial wealth is the same. The portfolio weights do not depend on the elasticity or discount parameters; only the time-0 consumption decision does.

From the budget constraint, the cost of time-1 consumption, \( \sum q_s c_s \), is equal to the time-0 expenditure, \( W_0 - c_0 \) so \( K \) can be expressed in terms of the amount invested

\[ \Psi_0 \equiv \rho^{1/\gamma} \psi_0^{1/\gamma} \] is permissible because the aggregator is ordinal.
\[ W_0 - c_0 = \sum_s q_s c_s = K \sum_s q_s^{1-1/\gamma} \pi_s^{1/\gamma} \Rightarrow c_s = \frac{(q_s/\pi_s)^{-1/\gamma}(W_0 - c_0)}{\sum_s q_s^{1-1/\gamma} \pi_s^{1/\gamma}}. \]  

(13)

To determine the time-0 consumption, substitute \( c_0^{p-1} \) for \( \lambda \) in the second relation in (12) then multiply both sides by \( c_s \), and sum across states

\[ \delta(\sum_s \pi_s c_s^{1-\gamma})^{p/(1-\gamma)} = c_0^{p-1} \sum_s q_s c_s = c_0^{p-1}(W_0 - c_0) \]  

(14)

where the final equality follows from the budget constraint, then substitute for \( c_s \) from (13) and solve for \( c_0 \)

\[ c_0^{p-1}(W_0 - c_0) = \delta(\sum_s \pi_s c_s^{1-\gamma})^{p/(1-\gamma)} = \delta \left[ \left( \frac{W_0 - c_0}{\sum_s q_s^{1-1/\gamma} \pi_s^{1/\gamma}} \right)^{p/(1-\gamma)} \sum_s q_s^{1-1/\gamma} \pi_s^{1/\gamma} \right]^{p/(1-\gamma)} \]

\[ \Rightarrow c_0^{p-1}(W_0 - c_0)^{1-p} = \delta \left[ \left( \sum_s q_s^{1-1/\gamma} \pi_s^{1/\gamma} \right)^{p/(1-\gamma)} \right]^{p/(1-\gamma)} \]

\[ \Rightarrow c_0 = \left[ 1 + \delta^{1/(1-p)} \left( \sum_s q_s^{1-1/\gamma} \pi_s^{1/\gamma} \right)^{p/(1-\gamma)(1-p)} \right]^{-1} W_0. \]  

(15)

This formula is the similar to that given in (10) for a CRRA investor, but there the sum was not raised to a power. When \( \gamma = 1 - \rho \), EZ preferences are time-additive CRRA, and the exponent here is then 1.

Again it is informative to consider the roles of risk aversion and elasticity of substitution. This can be illustrated in two special case. First, suppose there is only a single state at time 1; that is, there is no risk. In this case \( \pi = 1 \) and \( q = 1/(1 + r_f) \). Using (9) and (10) for time-additive CRRA utility,

\[ c_0 = \frac{W_0}{1 + \delta^{1/\gamma}(1+r_f)^{-1+1/\gamma}} \]

\[ c_1 = \frac{W_0[\delta(1+r_f)]^{1/\gamma}}{1 + \delta^{1/\gamma}(1+r_f)^{-1+1/\gamma}}. \]  

(16)

Obviously, time-0 consumption is increasing in \( \delta \) (decreasing in the rate of time preference). As there is no risk, \( \gamma \) is relevant only as a measure of elasticity of substitution, which is \( 1/\gamma \). As \( \gamma \) approaches \( \infty \), the investor’s EISC goes to zero, and consumption is smoothened as much as possible with both \( c_0 \) and \( c_1 \) approaching \( (1 + r_f)W_0/(2 + r_f) \). As \( \gamma \) approaches 0, the investor’s tastes become infinitely elastic, and he has no need of smoothing consumption. He either consumes all his wealth at time 0 or saves it all for time-1 consumption. He consumes at time 0 if \( \delta(1 + r_f) < 1 \) because then the return on saving is less that is rate of time preference. If \( \delta(1 + r_f) > 1 \), the benefit of saving is larger so he consumes nothing at time 0. Under unitary EISC (\( \gamma = 1 \)), the investor consumes the fraction \( (1 + \delta)^{-1} \) of his wealth at time 0 regardless of the interest rate.

The last case is general and does not depend on the absence of risk. For logarithmic CRRA utility (\( \gamma = 1 \)) or EZ preferences with an EISC of one (\( \rho = 0 \)), the average (and marginal) propensity to consume is \( c_0/W_0 = 1/(1 + \delta) \), completely independent of the state prices and probabilities. This is a property of the elasticity not the risk aversion; it is true for EZ preferences for any risk aversion. The independence of time-0 consumption from the state prices and probabilities also holds in incomplete markets and multi-period models. All of this makes utility with unit EISC, including time-additive logarithmic utility a particularly simple functional form to use in models. At the same time, those models may give results that do not extend to more general cases so some care must be used in interpretations.

**State Prices, the SDF and Risk-Neutral Probabilities**

In many pure-exchange models of equilibrium, it is assumed that consumption is known
and the objective is to determine the equilibrium prices. For example, it is commonly assumed that there is a representative agent with a known utility function, and that the distribution of per capita consumption is known. This problem can often be solved even when the portfolio problem is difficult to solve analytically.

Equating \( \lambda \) in the last two expressions in (2) gives the state prices as

\[
q_s = \lambda^{-1} \pi_s \frac{\partial U(c_0, c_s, s)}{\partial c_i} = \frac{\pi_s \partial U(c_0, c_s, s)}{\mathbb{E}[\partial U(c_0, c_s, \bar{s})/\partial c_0]} \quad \text{or} \quad q_s = \frac{\pi_s u'_i(c_s; s)}{u'_0(c_0)}. \tag{17}
\]

The latter is the simplification for time-additive utility. These solutions identify the supporting state prices from the investor’s probability beliefs and marginal utility. Because the state prices sum to \((1 + r_f)^{-1}\), time-0 consumption can be replaced giving

\[
q_s = \frac{\pi_s \partial U(c_0, c_s, s)}{(1 + r_f) \mathbb{E}[\partial U(c_0, c_s, \bar{s})/\partial c_0]} \quad \text{or} \quad q_s = \frac{\pi_s u'_i(c_s; s)}{(1 + r_f) \mathbb{E}[u'_i(\bar{c}_s; s)]}. \tag{18}
\]

The latter is for time-additive utility. These relations are particularly useful in models where there is no time-0 consumption. In such models, the state prices obviously cannot be expressed in terms of time-0 utility. Using the interest rate solves this problem. However, it is primarily a matter of calibration. If there is no first-period numeraire, the interest rate can only be given exogenously.

For example, for state-independent, time-additive CRRA utility

\[
q_s = \frac{\pi_s u'_i(c_s; s)}{u'_0(c_0)} = \frac{\pi_s \delta c^{-\gamma}}{c_0^{-\gamma}} \quad \text{or} \quad q_s = \frac{\pi_s c_s^{-\gamma}}{(1 + r_f) \sum_s \pi_s c_s^{-\gamma}}. \tag{19}
\]

Similar results obtain when utility is not von Neumann Morgenstern. From the first-order conditions in (12) for EZ preferences, the supporting price for state \( s \) is

\[
q_s = \frac{\delta(\sum_s \pi_s c_s^{-\gamma})^{-\gamma} \pi_s c_s^{-\gamma}}{c_0^{-\gamma}} \quad \text{or} \quad q_s = \frac{\pi_s c_s^{-\gamma}}{(1 + r_f) \sum_s \pi_s c_s^{-\gamma}}. \tag{20}
\]

Again the second equality follows because the state prices must sum to the risk-free discount factor. The second formula is identical to the state prices under CRRA utility in (19) because only \( \gamma \) affects attitudes about risk. However, the equality of the state prices in (19) and (20) does depend on the equality of the interest rates in the two economies, and the elasticity parameter \( \rho \) affects the trade-off between time-0 and time-1 consumption determining the interest rate in part. So with a given set of endowments in the two economies with CRRA and EZ investors, the interest rate will typically differ as a comparison of the first formulas in (19) and (20) shows.

In all of the formulas in (17) through (20), the state prices are proportional to the state probability, so the realizations for the SDF are the remaining factor on the right-hand side once the state probability is divided out. Risk-neutral probabilities are the state prices renormalized to sum to one

\[
m_s = q_s = \frac{u'_i(c_s; s)}{u'_0(c_0)} \quad \pi_s = q_s = \frac{u'_i(c_s; s)}{\mathbb{E}[u'_i(\bar{c}_s; s)]}. \tag{21}
\]

Using the relations in (18) instead

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3 This result is not true with more than two periods. In a multi-period problem, \( u_1 \) includes continuation utility, and both risk aversion and intertemporal elasticity affect risk premiums at time 0.
Again, these results are useful in models with no time-0 consumption. All of the results presented here are from the viewpoint of a given investor. It is the individual utility function, consumption, and beliefs that matter. However, as markets are complete, the state prices, which can be derived solely by the absence of arbitrage, must be the identical for all. Comparing the first-order conditions for two investors across two state, we have

$$\frac{\pi^k_s u_s^k(c^k_s)}{\pi^k_a u^k_a(c^k_a)} = \frac{q_a}{q_s}, \quad \frac{\pi^s_s u_s^s(c^s_s)}{\pi^s_a u^s_a(c^s_a)} = \frac{q_a}{q_s}. \quad (23)$$

Because the state prices are uniquely determined by the absence of arbitrage in a complete market, the middle ratio must be the same for both investors. So differences in investors’ beliefs must be offset by differences in their marginal utilities. If two investors with different beliefs have the same utility functions, they must consume different amounts in the various states; the investor with the higher subjective probability for state $s$ consumes more than an investor with the same utility who has a smaller subjective probability for that state.

The no-arbitrage asset pricing result is

$$p_n = \sum q s X_s = (1 + r_f)^{-1} \sum \hat{\pi}_s X_s = (1 + r_f)^{-1} \overline{E}[\hat{x}_s]. \quad (24)$$

The value of each asset is the risk-neutral expectation of its payoff discounted at the risk-free rate. This representation should be contrasted with the more standard pricing result in which the natural expectation is discounted at the interest rate plus a risk premium, $p_n = \overline{E}[\hat{x}_s] / (1 + r_f + k_\pi)$. This representation is the usual basis for the pricing of options and other derivative contracts.

The pricing relation can also be expressed as

$$p_n = \sum q s X_s = \sum \frac{\pi_s \partial U(c_0, c_s, s)/\partial c_1}{\overline{E}[\partial U(c_0, \hat{c}_1, s)/\partial c_0]} X_s = \overline{E}[\hat{x}_s \cdot \partial U(c_0, c_s, s)/\partial c_1] / \overline{E}[\partial U(c_0, \hat{c}_1, s)/\partial c_0], \quad (25)$$

or for time-additive utility

$$p_n = \sum q s X_s = \sum \pi_s [u'(c_s, s)/u'(c_0)] X_s = \overline{E}[\hat{x}_s u'_s(\hat{c}_1)] / u'_0(c_0). \quad (26)$$

**The SDF and the Growth-Optimal Portfolio**

The marginal utility description for the SDF is particularly useful with time-additive, state-independent, logarithmic utility. In this case (22) is

$$m_s = \frac{u'_s(c_s)}{u'_0(c_0)} = \frac{\delta c_0^{-1}}{c_s} = \frac{\delta c_0}{c_s}. \quad (27)$$

From (10), $c_0 = W_0 / (1 + \delta)$ so $\delta c_0 = W_0 - c_0$. Substituting into (27) gives

$$m_s = \frac{\delta c_0}{c_s} = \frac{W_0 - c_0}{c_s} \equiv \frac{1}{1 + r_f^n} \quad (28)$$

where $r_f^n$ is the rate of return on the portfolio that is optimal for an investor with time-additive utility.

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4Note that equation (25) is not $p_n = \overline{E}[\hat{x}_s \cdot \partial U(c_0, c_s, s)/\partial c_1, \partial U(c_0, c_s, s)/\partial c_0] / \overline{E} X_s$ in general. It can be, and often is, written this way when utility is time additive, because the denominator is then not random.
logarithmic utility. The last equality follows because the investor consumes his entire remaining wealth in period 1, so \( c_s \) is the gross return realized in state \( s \) on the portfolio optimal for log utility. For historical reasons, this portfolio is known as the growth-optimal portfolio.

So the SDF is the reciprocal on the return on the growth-optimal portfolio. As with many other special properties of log utility, this property extends to incomplete markets and multi-period models as we will see later.

**Risk Aversion, Portfolio Riskiness, and Expected Returns**

It would seem obvious that more risk averse investors would hold portfolios that are less risky, which therefore have lower expected rates of return. For convenience, the discussion below is phrased in terms of time-1 consumption rather than portfolio returns. All the results apply to the portfolios of investors who commit the same total amount, because the portfolio return is 1 + \( \tilde{r}_k = \tilde{c}_k/(W_{0k} - c_{0k}) \).

To illustrate, suppose there are only two CRRA or EZ investors with relative risk aversions of \( \gamma_1 \) and \( \gamma_2 < \gamma_1 \) and homogeneous beliefs. For each investor, \( c_k^{\gamma} = \lambda_k q_k / \pi_k \). Therefore,

\[
\frac{\tilde{c}_1^{\gamma_1}}{\lambda_1} = \frac{\tilde{c}_2^{\gamma_2}}{\lambda_2} \Rightarrow \left( \frac{\lambda_2}{\lambda_1} \right)^{1/\gamma_2} \tilde{c}_1^{\gamma_1/\gamma_2} + \tilde{c}_1 = C^{\text{agg}}.
\]

The value of the lead coefficient depends on the relative wealth of the two investors, their risk aversions, and the sets of state prices and probabilities. However, we can still determine the general relation. The first investor’s consumption is increasing in aggregate consumption; as is the second investor’s by similar reasoning. When \( C^{\text{agg}} \) is very small so is \( c_1 \), and the lead term is negligible because the exponent is greater than one. So the more risk averse investor receives nearly all of the consumption. When \( C^{\text{agg}} \) is large so is \( c_1 \), so the first term dominates. In the very best states as \( C^{\text{agg}} \) grows, \( c_1 \) grows only in proportion to \( (C^{\text{agg}})^{\gamma_1/\gamma_2} \) and the more risk tolerant investor receives nearly all of the consumption.

However, this intuition is not correct. Dybvig and Wang (2012) provide a counterexample. Two investors have the utilities \( u_1(c) = -(8 - c)^{-3}/3 \) and \( u_2(c) = -(8 - c)^{-5}/5 \) for \( c < 8 \). The absolute risk aversions are \( A_1(c) = 2/(8 - c) < A_2(c) = 4/(8 - c) \) so investor 2 is more risk averse. They each have a wealth of 5 to invest. The market and their optimal portfolios are given in the table below.

<table>
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<th>state</th>
<th>( \pi )</th>
<th>( q )</th>
<th>( c_1 )</th>
<th>( u_1' )</th>
<th>( \pi u_1'/q )</th>
<th>( c_2 )</th>
<th>( u_2 )</th>
<th>( \pi u_2'/q )</th>
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<td>2.1011</td>
<td>7.99988</td>
<td>1.39710</td>
<td>3.4932</td>
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</tr>
<tr>
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<td>0.3125</td>
<td>5.10369</td>
<td>4.36658</td>
<td>3.4932</td>
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<td>31.4419</td>
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</tr>
<tr>
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<td>0.6250</td>
<td>5.044815</td>
<td>8.73315</td>
<td>3.4932</td>
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</tr>
</tbody>
</table>

The cost of each portfolio is 5 so they are feasible. The ratios \( \pi u_k'(c_k)/q \) are equal across states for both investors so each portfolio is optimal. The expected consumption and variances are \( \mathbb{E}[\tilde{c}_1] = 6.318 > \mathbb{E}[\tilde{c}_2] = 6.270 \) and \( \text{var}[^{\tilde{c}_1}] = 1.684 < \text{var}[^{\tilde{c}_2}] = 1.697 \). Because \( \tilde{c}_1 \) has a smaller variance than \( \tilde{c}_2 \), it cannot be Rothschild-Stiglitz riskier. So the intuition stated at the beginning of this section cannot be correct.

Although \( \tilde{c}_1 \) is not necessarily riskier than \( \tilde{c}_2 \), the two investors’ consumptions can be described by \( \tilde{c}_1 = d \tilde{c}_2 + \tilde{\eta} + \tilde{\epsilon} \) with \( \tilde{\eta} \geq 0 \) and \( \mathbb{E}[\tilde{\epsilon} | \tilde{c}_2 + \tilde{\eta}] \). When \( \tilde{\eta} + \tilde{\epsilon} \) is negatively correlated with \( \tilde{c}_2 \), it is possible that \( \text{var}[\tilde{c}_1] < \text{var}[\tilde{c}_2] \) as in the example.

**Theorem 4.2: Increased Risk Aversion and Optimal Portfolios.** In a complete market, if two

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5 The random variables \( \tilde{\eta} \) and \( \tilde{\epsilon} \) are not necessarily unique.
investors invest the same amount, have homogeneous beliefs, and the first investor is strictly less risk averse than the second in the Arrow-Pratt sense, then for any set of no-arbitrage state prices,

i) \( \mathbb{E}[\tilde{c}_1] > \mathbb{E}[\tilde{c}_2] \)

ii) \( \tilde{c}_1 = \tilde{c}_2 + \tilde{\eta} + \tilde{\varepsilon} \) where \( \tilde{\eta} \geq 0 \) and \( \mathbb{E}[\tilde{\varepsilon} | c_2 + \eta] = 0 \).

**Proof:** If investor \( \kappa \) is strictly more risk averse than investor \( k \), and both investors have twice-differentiable utility functions, then from Theorem 2.4 \( u_k'(c) = G'(u_k(c)) \) with \( G' > 0 \) and \( G'' < 0 \).

Marginal utility of investor \( \kappa \) is \( u_k'(c) = G'(u_k(c))u_k(c) \). Define the SDF to be the random variable with realizations \( m_k \equiv q_k^0/\pi_k \). Then the first order conditions for investors 1 and 2 are related state-by-state as

\[
\lambda_1 m_1 = \lambda_2 m_2 = \frac{\lambda_1}{\lambda_2} u_1'(\tilde{c}_2) = \frac{\lambda_1}{\lambda_2} G'(u_1(\tilde{c}_2))u_1'(\tilde{c}_2). \tag{30}
\]

Because both marginal utilities and \( G' \) are strictly decreasing, \( \tilde{c}_1 \geq \tilde{c}_2 \) when \( G'(u_1(\tilde{c}_2)) \leq \lambda_2 / \lambda_1 \). Define \( c^o \) as the equality point; i.e., \( G'(u_k(c^o)) \equiv \lambda_2 / \lambda_1 \). Then in states with low consumption where \( \tilde{c}_2 < c^o \), \( G'(u_k(c^o)) \lambda_1 / \lambda_2 < 1 \), and from (30) \( u_1'(c_1) < u_2'(c_2) \). As marginal utility is decreasing, \( c_1 > c_2 \) in those states. Similarly when \( \tilde{c}_2 > c^o \), \( \tilde{c}_2 > \tilde{c}_1 \). This means that the cumulative distribution functions of \( \tilde{c}_1 \) and \( \tilde{c}_2 \) cross exactly once.

As the market is complete, the SDF \( \tilde{m} \) is unique and inverse monotonically related to both consumptions. From the first-order condition, \( \lambda_2 m_2 = \lambda_2 u_2'(\tilde{c}_2) = \frac{\lambda_1}{\lambda_2} G'(u_1(\tilde{c}_2))u_1'(\tilde{c}_2) \). So

\[
0 = \mathbb{E}[\tilde{m}(\tilde{c}_1 - \tilde{c}_2)] = \mathbb{E}[m^o(\tilde{c}_1 - \tilde{c}_2)] + \mathbb{E}[(\tilde{m} - m^o)(\tilde{c}_1 - \tilde{c}_2)] < m^o \mathbb{E}[\tilde{c}_1 - \tilde{c}_2]. \tag{31}
\]

The inequality follows because the random variable in the eliminated expectation is negative. The SDF value \( m^o \) must be positive so \( \mathbb{E}[\tilde{c}_1] > \mathbb{E}[\tilde{c}_2] \). This proves (i).

Now define new random variables \( \tilde{\omega}_1 \equiv -\tilde{c}_1 \). The cumulative functions for \( \tilde{\omega}_1 \) and \( \tilde{\omega}_2 \) must cross exactly once just as those for \( \tilde{c}_1 \) and \( \tilde{c}_2 \) do with the cumulative distribution for \( \tilde{\omega}_1 \) above that for \( \tilde{\omega}_2 \) at small values. Also \( \mathbb{E}[\tilde{\omega}_1] = -\mathbb{E}[\tilde{c}_1] < -\mathbb{E}[\tilde{c}_2] = \mathbb{E}[\tilde{\omega}_2] \). This means that \( \tilde{\omega}_2 \) second-order stochastically dominates \( \tilde{\omega}_1 \). Therefore, \( \tilde{\omega}_1 = \tilde{\omega}_2 + \tilde{\eta}' + \tilde{\varepsilon}', \) with \( \tilde{\eta}' \leq 0 \) and \( \mathbb{E}[\tilde{\varepsilon}' | \omega_2 + \eta] = 0 \), or

\[
\tilde{c}_1 = \tilde{\omega}_1 = -\tilde{\omega}_2 - \tilde{\eta}' - \tilde{\varepsilon}' = \tilde{c}_2 + \tilde{\eta} + \varepsilon, \tag{32}
\]

with \( \tilde{\eta} \equiv -\tilde{\eta}' \geq 0 \) and \( \mathbb{E}[\tilde{\varepsilon}' | c_2 + \eta] = -\mathbb{E}[\tilde{\varepsilon}' | \omega_2 + \eta] = 0 \). This proves (ii).

Dybvig and Wang also prove that a sufficient condition for \( \tilde{\eta} = \mathbb{E}[\tilde{c}_1 - \tilde{c}_2] \), the only possible constant value, is that either of the two utility functions has non-increasing absolute risk aversion. In this case, \( \tilde{c}_1 \) is riskier that \( \tilde{c}_2 \).

The extreme numbers in the example would seem to indicate that a more risk averse investor holds a less risky portfolio except under unusual circumstances. This intuition is probably correct. Unfortunately, it is difficult to make a precise definition of “unusual”. The example is presented mostly as a precaution. In many cases, an intuitive result can be supported by numer-

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6 The second inequality is strict because the risk aversion comparison is strict.

7 Define \( c^o \equiv \pm \infty \) if \( G'(u_k(\tilde{c}_2)) \geq \lambda_2 / \lambda_1 \) everywhere. This case must be considered here as the budget constraints have not yet been applied so either consumption variable could dominate the other.

8 To be precise, the dominance is single-crossing stochastic dominance as defined in the Further Notes section of Chapter 2.
ous examples with no counterexamples. In modelling, care must be exercised to avoid making incorrect assumptions as they can lead to “theorems” that are contradictory.⁹

**Equilibrium in a Complete Endowment Market**

The previous analysis derived investors’ optimal portfolios and showed how the prices of assets are related to the state prices, but it did not show how prices are determined from fundamentals. That is the role of equilibrium models; they derive the underpinnings for the results this far.

To create an equilibrium, the economy must adjust until markets clear and each investor holds an optimal portfolio subject to whatever natural or man-made constraints are imposed. The adjustments can be in supplies or prices or both.

In Finance, the type of economy most often examined is the pure-exchange or endowment economy. In these economies, the physical investment decisions have already been made. The equilibrium is achieved by the adjustment of prices through the actions of the investors together. It is generally assumed that each consumer is small enough so as not to affect prices through trading; that is, all investors are price takers.

To set up the economy, each investor is endowed with the existing assets either by assumption or as the result of a previous unmolded period in the economy. When markets are complete, we can express the endowments and final portfolios in terms of pure state securities. Investor $k$’s endowment is denoted $\eta_k$. The total or market endowment is $\sum_k \eta_k$. A competitive equilibrium then is a portfolio choice for each investor $\eta_k$ and a set of state prices $q$ such that markets clear, $\sum_k \eta_k = \sum_k \bar{\eta}_k$ and each investor holds his optimal portfolio

$\eta_k = \arg\max \mathbb{E}[u_k(\tilde{c}_k)]$ subject to $c_{sk} = \eta_{sk}$ $q^k \eta_k = q^k \bar{\eta}_k$

(33)

Because first-period consumption has been ignored here, only relative state prices are defined. The normalization $\mathbf{1}'q = 1$ will be used. This equates the state prices and the risk-neutral probabilities.

Determining the equilibrium involves solving for $S \times (K+1)$ variables in a set of nonlinear equations. The state price and the holdings of each of $K$ investors must be determined. Except under special conditions, analytical solutions cannot be found with heterogeneous investors. Some small solved examples follow. In each case, investor 1 has logarithmic utility with a risk aversion of 1, and investor 2 has square root utility with a risk aversion of $\frac{1}{2}$. The endowments are given in the first column panel. In the second panel, the two investors have homogeneous beliefs that the two states are equally likely. In the third panel, investor 1 thinks state 1 is half as likely while investor 2 thinks state 1 is twice as likely.

Several features are obvious in these examples. When the investors have homogeneous beliefs, they always hold portfolios that are perfectly aligned; they each have more consumption in a state with higher aggregate consumption available.¹⁰ This property will be discussed in more depth later. In the risk-free economy their consumptions are equal in each state. This observation would remain true even if the states had different probabilities. But it is obviously not true under heterogeneous expectations.

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⁹ In Boolean logic, the truth value of “false implies true” is true, so in principle a false assumption can prove any statement, true or false.

¹⁰ In this case with only two states, the portfolios are also perfectly correlated, but that is only because there are just two states so perfect alignment must be perfect correlation. This is also why the expected rates of return and standard deviations are in the same ratio.
When the investors have homogeneous beliefs, investor 1, who is twice as risk averse, takes on less risk than investor 2; the spread in his payoffs is half as large. He “pays” investor 1 to provide this service by allowing him to have a higher expected rate of return that is twice as large. (In general it is the excess expected rate of return that would be twice as large, but here the interest rate has been arbitrarily set to zero.) Standard deviation is a valid measure of risk in this simple two equally probable state economy. It general, that investor 2’s portfolio would be riskier and have a higher return, though standard deviation would not be the proper measure of risk.

When both investors’ endowments double, their consumption doubles in each state while the state prices remain the same. This is a special property of CRRA utility. It is also true with heterogeneous beliefs, and we will see later, it remains true when markets are incomplete. When the endowment of just one investor doubles, his portfolio holdings do not double because the change in the relative allocation of the endowments between investors changes the state prices. This affects the returns distribution and alters the portfolios further. Investor 1 is more risk averse so when his endowments increase, the total demand for safety rises. This increases the state price for the bad state 1 relative to that for the good state 2. Conversely when the endowment of investor 2 increases, the state price for state 2 rises while state 1’s price drops.

### Examples of Equilibria in Pure-Exchange Economies

Utility functions for investors are: \( u_1(c) = \ell \ln(c) \), \( u_2(c) = \sqrt{c} \)

<table>
<thead>
<tr>
<th>Endowments</th>
<th>( \pi_1 = \pi_2 = (\frac{1}{2}, \frac{1}{2})' )</th>
<th>( \pi_1 = (\frac{1}{3}, \frac{2}{3})' ), ( \pi_2 = (\frac{2}{3}, \frac{1}{3})' )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Holdings</td>
<td>State Price</td>
</tr>
<tr>
<td></td>
<td>Inv 1</td>
<td>Inv 2</td>
</tr>
<tr>
<td>Risk-free economy</td>
<td>( \bar{\eta}_1 )</td>
<td>( \bar{\eta}_2 )</td>
</tr>
<tr>
<td>Low equal endowments</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{1}{2} )</td>
</tr>
<tr>
<td>High equal endowments</td>
<td>( \frac{2}{4} )</td>
<td>( \frac{2}{4} )</td>
</tr>
<tr>
<td>Investor 2 wealthier</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{1}{2} )</td>
</tr>
<tr>
<td>Investor 1 wealthier</td>
<td>( \frac{2}{4} )</td>
<td>( \frac{1}{2} )</td>
</tr>
</tbody>
</table>

This does not characterize the portfolios under heterogeneous beliefs. In those cases, each investor increases consumption in the state he views as more likely. At times this is sufficient to make investor 2’s consumption higher in the state with smaller aggregate consumption. Also investor 1 now holds the riskier portfolio with the higher expected rate of return. He does this because he is more optimistic than investor 2 attaching the higher probability to the state with the larger payoff. With a lesser disagreement between the two investors, investor 2 would still hold a riskier portfolio. “Riskier” here is based on the “correct” equally probable states. Investor 1 does not think his portfolio is riskier using his own probabilities.
Another common equilibrium model is a production economy typically with stochastic constant returns to scale processes. In this economy, the production processes can be characterized by a set of random variables, $\tilde{z}_i$. An input of $k$ units to the process produces an output $\tilde{x}_i = k\tilde{z}_i$. The production processes can be correlated, but there are no economies of scope so that the input to one process has no effect on the output from another.

This economy can be represented as one in which the processes are all assets with a price 1; then each column of the payoff matrix, $X$, is the state-by-state realizations on one process. If this matrix has full rank, then the market is complete and the state prices satisfy $X'q = 1$. Each investor can solve his own portfolio problem to determine, $q^*$, and in equilibrium, the total supply of the assets is $\sum q^*$. If all the $q^*$ are nonnegative, this creates an economy known as an autarky as each investor is sufficient unto himself, and no trading need take place.

It becomes more difficult if a component of $q^*$ is negative because negative physical investment is not possible. As an example consider the following economy. There are two production processes, the first is risk-free and returns 2 units for every unit input. The second process returns either 1 or 4 units with probabilities 1/3 and 2/3, respectively. An investor with utility $u_1(c) = -1/c$ will optimally invest in the two processes equally. An investor with utility $u_2(c) = \sqrt{c}$ is willing to take more risk and would like to invest in proportions $-2/3$ and $5/3$.

If only investors of the first type are present, then both processes will be used equally. The economy will produce in aggregate either 1.5 or 3 units for every input unit available, and the state prices will be 1/3 and 1/6. These state prices can be determined from the production processes themselves

$$q_1 + 4q_2 = 1 \quad \text{and} \quad 2q_1 + 2q_2 = 1 \implies q_1 = \frac{1}{3}, \ q_2 = \frac{1}{6}.$$  \hspace{1cm} (34)

The relative state prices can be determined from the first order conditions, $\pi_iu'(c_i) = \lambda q_i$,

$$\frac{q_1}{q_2} = \frac{\pi_1u'(1.5)}{\pi_2u'(3)} = -\frac{1}{3}(1.5)^{-2} \quad \frac{-\frac{2}{3}3^{-2}}{2} = 2.$$  \hspace{1cm} (35)

If only investors of the second type are present, they cannot make a negative investment in the safe process as they would wish, so only the risky process will be used in equilibrium. The economy will produce either 1 or 4 units for each unit input. The market will be incomplete in terms of the production processes. Investors could introduce a financial asset to complete the market, but it would have to be in zero net supply.

The subjective state prices for the investors must price the one process used correctly, $q_1 + 4q_2 = 1$, and must satisfy the first-order conditions for the constrained optimization $\pi_iu'(\tilde{x}_i) = \lambda q_i$. Eliminating the Lagrange multiplier by taking the ratio we have

$$q_1 + 4q_2 = 1 \quad \text{and} \quad \frac{\frac{1}{3}\sqrt{1}}{\frac{2}{3}\sqrt{4}} = \frac{q_1}{q_2} \implies q_1 = \frac{1}{17}, \ q_2 = \frac{4}{17}.$$  \hspace{1cm} (36)

No one will borrow or lend, but the shadow interest rate is $1/(q_1 + q_2) - 1 = 17/5 - 1 = 240\%$ rather than the 100% that is implied by the unused safe process. The interest rate needs to be this high to keep the net demand for borrowing at zero because none of the investors wishes to lend.

With both types of investors, the resulting equilibrium depends on how many of each are present, or more precisely, on the relative amount of capital each type has to invest. Suppose there

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1 Because both investors have CRRA utility, their optimal investment proportions do not depend on the amount of input they have available. With more wealth they simply scale up their investment.
are 9 investors of second type and 20 investors of first type each with a single unit of capital. The second type investors each wish to invest $\frac{5}{3}$ in the risky process and $-\frac{2}{3}$ in the safe process for a total of $15$ and $-6$. The first type wants to invest a total of $10$ units in each process. So the desired total investments for the 29 units available are $25$ and $4$.

To accomplish this, type 2 investors borrow 6 units from the others promising to return 12. This is the same 100% interest rate the safe process promises, and will be default-free as shown below, so the type 1 investors are satisfied to do so. The 6 units borrowed plus the 9 units the type 2 investors own are invested in the risky process. Type 1 investors have invested 6 units in bonds. Their remaining 14 units are split with 4 in the safe process and 10 in the risky process. This gives them 10 units invested safely and 10 invested in the risky process as desired.

If the bad state occurs, type 2 investors realize 15 on their investment and repay 12 leaving themselves with 3 units or $\frac{1}{3}$ unit each. Type 1 investors receive 12 from the loan repayment and 8 from their safe investment. Their risky investment returns 10. So they have 30 total or $\frac{1}{3}$ each.

If the good state occurs, type 2 investors realize 60 and have 48 units left after paying off the debt. This is $\frac{16}{3}$ each. The type 1 investors receive 12 from their loan repayment, 8 from their safe investment and 40 from their risky investment. The total payout of 60 is 3 units each.

The debt can be paid in either state so it is risk-free just like the production process meaning the investors are indifferent to lending or investing on their own. Both processes are being used so the state prices must be $\frac{1}{3}$ and $\frac{1}{6}$ as determined by the processes themselves just as in the first case considered. The two-to-one ratio of the state prices is also verified by the first order conditions

$$\frac{q_1}{q_2} = \frac{\pi_k u'_1(1.5)}{\pi_k u'_1(3)} = \frac{-\frac{1}{3}(1.5)^{-2}}{-\frac{2}{3}3^{-2}} = 2 \quad \frac{q_1}{q_2} = \frac{\pi_k u'_1(1/3)}{\pi_k u'_1(16/3)} = \frac{\frac{1}{3} \frac{1}{3}^{-1/2}}{\frac{2}{3} \frac{16}{3}^{-1/2}} = 2.$$  \hspace{1cm} (37)

**Risk Sharing and Pareto Optimality**

Consider two investors, $k$ and $\kappa$. From (17), their consumption in any two states $s$ and $s'$ are related by

$$\frac{\pi_k}{\pi_\kappa} \frac{\partial U_k(c_k^s, c_\kappa^s, s)}{\partial c_1} = \frac{q_k}{q_\kappa} = \frac{\pi_\kappa}{\pi_\kappa} \frac{\partial U_\kappa(c_\kappa^s, c_\kappa^s, s)}{\partial c_1}.$$

(38)

If they have homogeneous beliefs, then their marginal utilities are perfectly correlated across states. This condition is known as perfect risk sharing. Note it is marginal utility that is perfectly correlated rather than utility or consumption. It is perfect risk sharing because there are no remaining idiosyncratic variations in utility that can be altered at the margin.

Perfect risk sharing is a special case of Pareto optimality (or Pareto efficiency) which holds in a complete market even with heterogeneous beliefs. A Pareto optimal allocation of the total available consumption, $\hat{C}$, is one for which there is no other allocation that gives any investor a higher expected utility without decreasing the utility of some other investor. By definition, this must be true even for allocations that violate some investor’s budget constraint. The only constraint on Pareto allocations is global feasibility.

The allocation achieved by price-taking investors trading in a complete market is Pareto optimal. In general there are many allocations that are Pareto optimal even though the complete-market allocation in equilibrium is unique. In additional Pareto optimality says nothing about fairness. Giving all the time-1 consumption to one investor is Pareto optimal, because every other

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12 They can either do this individually or form a company to do the borrowing and investing. As the processes have stochastic constant returns to scale both methods are equivalent.
allocation would make him worse off (assuming no satiation); though it surely is not “fair”.

If all investors have strictly increasing utility, a $K$-vector allocation $\tilde{c}^o$ is Pareto optimal if it satisfies the aggregate budget constraint $1^T \tilde{c}^o \equiv \tilde{C}^\text{agg}$, and

$$E_k[u_k(\tilde{c}_k)] \leq E_k[u_k(\tilde{c}_k^o)] \quad \forall k.$$  \hfill (39)

Pareto-optimal allocations can be found as solutions to $K$ simultaneous Lagrangian problems

$$\max_{\tilde{c}_k} \alpha_k \sum_s \pi_s^k u_k(c_{sk}) + \sum_s \gamma_s \left( C^\text{agg}_s - \sum_k c_{sk} \right) + \sum_{k,s,k} \alpha_k \pi_s^k \left[ u_k(c^o_{sk}) - u_k(c_{sk}) \right]$$

Each of these $K$ problems looks for the maximum utility that one investor can be given subject to the constraints that the expected utility of every other investor is at the same level as in the presumed Pareto optimal allocation, $\tilde{c}^o$, and that all the consumption available in each state is allocated to some investor. Equality constraints can be used in both cases here because the investors are assumed to have strictly increasing utility and cannot be satiated so investor $k$’s utility could always be increased by reallocation to him any unallocated consumption or some consumption assigned to an investor whose expected utility exceeded that in the $\tilde{c}^o$ allocation. The first-order conditions are

$$\alpha_k \pi_s^k u^*_k(c^*_k) = \gamma_s \quad \forall s, k \quad \sum_k c^*_sk = C^\text{agg}_s \quad \forall s \quad E_k[u_k(c^*_k) - u_k(c^o_{sk})] = 0 \quad \forall \kappa \neq k.$$  \hfill (41)

These problems and solutions that define Pareto optimality can also be analyzed as the problem faced by a benevolent central planner seeking to maximize a weighted average of the investors’ expected utility subject to the total allocation constraint. The Lagrangian problem and first-order conditions are

$$\max_{\tilde{c}_k} \omega_k \sum_s \pi_s^k u_k(c_{sk}) + \sum_s \beta_s \left( C^\text{agg}_s - \sum_k c_{sk} \right)$$

$$\Rightarrow \omega_k \pi_s^k u^*_k(c^*_k) = \beta_s \quad \forall s, k \quad \text{and} \quad \sum_k c^*_sk = C^\text{agg}_s \quad \forall s$$

for some arbitrary set of positive weights $\omega_k$. In any given problem, $\omega_k$ is a measure of how important investor $k$’s utility is in the central planner’s problem. Different choices for $\omega$ map out all possible Pareto optimal allocations.

Because the first order conditions are identical in (42) and (41), and the expected utilities are matched by construction, it is clear that any solution $c^{**}(\omega)$ cannot be beaten when chosen as the target Pareto-optimal solution $c^o$ in the original problem. So $c^{**}(\omega)$ is the Pareto optimal solution for weights $\omega$.

The individual budget constraints do not appear in the definition or solution of Pareto optimal allocations, nor do any of the state prices. Pareto optimal allocations are just that — allocations and not necessarily equilibrium solutions. However, it is a standard result in Economics that a competitive equilibrium in a complete market is Pareto optimal. This can be verified immediately by comparing the first order conditions in the central planner problem to those in the investor’s optimization, $\pi_s^k u^*_k(c_{sk}) = \lambda_s q_s$. Choosing weights so that $\beta_s / \omega_k = \lambda_s q_s$ identifies that optimal portfolio as one Pareto optimal allocation.

The first-order conditions to the central planer problem include those of the individual investors’ problems in a complete market. So that solution is also Pareto optimal. In fact, the competitive equilibrium is the unique Pareto optimal allocation for a given set of state prices. This can be verified directly for local changes in the allocations.

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13 Time-0 consumption can be included in an obvious fashion. The leading $\alpha_k$ in (40) is not required. It does not affect the maximization provided it is positive. It is used merely to keep the solutions symmetric.
Suppose investor $k$’s time-1 consumption is changed in some states by $dc_{sk}$. This alters his expected utility by

$$d\mathbb{E}_k[u_k(c_{0k}, \tilde{c}_{sk}, s)] = \sum_s \pi^k_s \hat{c}_k u_k(c_{0k}, c_{sk}, s) dc_{sk} = \lambda_k \sum_s q_s dc_{sk}$$

(43)

where the second equality follows from the first-order condition (2).\(^{14}\) The lagrange multiplier for each investor is positive so $\lambda_k d\mathbb{E}_k[u_k()] = \sum_s q_s dc_{sk}$. Summing over all investors we have

$$\sum_k \lambda_k d\mathbb{E}_k[u_k()] = \sum_s q_s \sum_k dc_{sk} = \sum_s q_s \cdot dC_s^{agg} = 0.$$  

(44)

The reallocation does not change total consumption in any state so $dC_s^{agg} \equiv 0$ in every state, and the final sum must be zero. As the Lagrange multipliers in the first sum are all positive, the first sum can be zero when some investor’s expected utility increases only if at least one other investor suffers a decrease in expected utility. Therefore, the reallocation cannot maintain a Pareto optimal allocation.

This proof only considers changes at the margin. However, for strictly concave utility, there can be no nonlocal improvements in expected utility unless a local improvement is possible.

**Effectively Complete Markets**

Unfortunately, markets would appear to be woefully incomplete. Nevertheless, the market may be *effectively complete* with agreement on state prices even in the absence of a truly complete market.

If all investors’ utility functions are state-independent, then the different states need be characterized only by the pattern of payoffs on the various assets. Note that these states are “bigger” than Arrow-Debreu states in which commodity prices, the weather, individuals’ health, etc. must also be distinguished. A market that is complete over this description of broader states is sometimes called a market of complete prices rather than a complete market. In a market of complete prices, the unique state price for every broadly-defined state can be determined from the prices of the actual assets by the absence of arbitrage alone. This need not be true in an Arrow-Debreu complete market where the actual asset prices may not contain enough information to even distinguish all the states.

If the market is not even complete in the complete-prices sense, then arbitrage alone cannot determine all the state prices. This is the most apt description of actual financial markets. There are many times more states than securities. Nevertheless, the complete market pricing results can often be extended to cover such markets.

Suppose an investor has determined his optimal portfolio and consumption pattern across the states subject to whatever restrictions the lack of a complete market imposes. From investor $k$’s optimal portfolio determine a set of individual or *subjective state prices* defined as

$$q^k_s = \frac{\pi^k_s \partial U_k(c_{0s}, c_{is}, s)/\partial c_{is}}{\mathbb{E}_k[\partial U_k(c_{0s}, \tilde{c}_{is}, \tilde{s})/\partial c_{0s}]}.$$  

(45)

Now solve this investor’s problem again assuming he is trading in a complete market with the state prices equal to the subjective prices determined in (45). Clearly he holds the same portfolio as in the constrained incomplete market.

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\(^{14}\) This is where strictly positive rather than merely weakly positive marginal utility is required. The first order condition need not hold for a satiated investor. Indeed, if in some investor is satiated at time 0 and in all states at time 1, then there are Pareto improvements that reallocate consumption to non-satiated investors. However, for this to occur the investor must be satiated in all states; otherwise he could improve his own expected utility by adjusting his consumption across states or time. In a complete market these adjustments are always possible.
Some of these subjective state prices might have the same value for every investor. This must be true for every insurable state for which an Arrow-Debreu state security can be constructed from the existing assets. For other states, the subjective state prices may differ across investors. If the subjective state prices are the same for all investors in all states, then the market is effectively complete because no trades that are disallowed by the constraints imposed by the existing assets would increase any investor’s expected utility. The fact that those trades are not allowed is not a binding constraint so we can pretend they are allowed — that is, pretend that the market is complete. Subjective state prices are obviously equal if all investors have the same utility function and beliefs, but even when investors are heterogeneous, we might expect equal subjective state prices in equilibrium.

Suppose there are two investors for which subjective state prices differ in some state with \( q^k_s < q^\kappa_s \). Then from (45)

\[
\frac{\pi^k_s \partial \bar{U}_k(c_0, c_s, s) / \partial c_1}{\bar{E}_k[\partial \bar{U}_k(c_0, \bar{c}_1, \bar{s}) / \partial c_0]} \equiv q^k_s < q^\kappa_s \equiv \frac{\pi^\kappa_s \partial \bar{U}_\kappa(c_0, c_s, s) / \partial c_1}{\bar{E}_\kappa[\partial \bar{U}_\kappa(c_0, \bar{c}_1, \bar{s}) / \partial c_0]}.
\]

(46)

Because investor \( \kappa \) has higher marginal utility in state \( s \) relative to time-0 marginal compared to investor \( k \), a trade in which investor \( k \) gives investor \( \kappa \) some state \( s \) consumption in exchange for some time-0 consumption can increase both of their expected utilities. If creating a derivative or financial asset that permits this trade is not prohibited and is not too costly, that market should always be opened — and this is true even if the new equilibrium makes other investors worse off.\(^{15}\) In the absence of frictions to create new securities, this process should continue until investors’ subjective state prices are equalized. Therefore, we should expect that the market will be effectively complete or, at least, that any differences in subjective state prices will be smaller than the cost of creating new financial securities. If the market is effectively complete and investors have homogeneous beliefs and state independent utility, then their consumptions will align across states just as in a complete market, and the market portfolio will necessarily be efficient.\(^{16}\)

The same is true in a model with no first-period consumption. With no first-period consumption, only relative subjective state prices are defined for each investor. If \( q^k_s / q^\kappa_s < q^\kappa_s / q^\kappa_s \), then both investors’ utility can be increased by transferring some state \( s \) consumption to investor \( \kappa \) in exchange for state \( s' \) consumption.

In fact, an effectively complete market is often called a Pareto-optimal market for exactly this reason. The market is effectively complete, if and only if the consumption allocation is Pareto optimal so that no feasible reallocation can improve one investor without hurting another. This must be true because when investors’ subjective state prices differ, there is always some reallocation that increases the expected utility of both so the original allocation could not have been Pareto optimal. The next section discusses the costs and benefits of completing the market.

### Completing the Market

If the market is not effectively complete, some investors can always be made better off when additional assets are created provided the costs of doing so are not too high. However, this

\(^{15}\) The Pareto optimality of a complete market does not guarantee that creating any security that moves the market closer to complete or even completes the market must be weakly Pareto improving. The next section provides an example.

\(^{16}\) There are several notions of efficiency in Finance. Portfolio efficiency, means that there is an increasing concave utility function (or a utility function in some other class) for which the portfolio is optimal. The previously mentioned Pareto optimality is sometimes called Pareto Efficiency. Informational efficiency will be introduced later.
does not mean that adding assets to complete the market necessarily makes all investors better off. This is examined in a few simple endowment markets below.

The table below outlines the first problem. There are two equally probable states next period, \( a \) and \( b \). There are two investors with time-additive, state-independent CRRA utility with relative risk aversions of \( \frac{1}{2} \), \( u(c) = c^{1/2} \). For simplicity the subjective discount rates are set to zero. The consumption good cannot be saved from time 0 to time 1.

<table>
<thead>
<tr>
<th>Inv.</th>
<th>( u(c) )</th>
<th>( u'(c) )</th>
<th>( \hat{q} )</th>
<th>( \hat{q} )</th>
<th>( \mathbb{E}[u] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Autarky</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>9</td>
<td>3.00</td>
<td>0.33</td>
<td>16</td>
<td>4.00</td>
</tr>
<tr>
<td>2</td>
<td>9</td>
<td>3.00</td>
<td>0.33</td>
<td>4</td>
<td>2.00</td>
</tr>
<tr>
<td>with trade</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1,2</td>
<td>9</td>
<td>3.00</td>
<td>0.33</td>
<td>10</td>
<td>3.16</td>
</tr>
<tr>
<td>1,2</td>
<td>8.05</td>
<td>2.84</td>
<td>0.35</td>
<td>10</td>
<td>3.16</td>
</tr>
</tbody>
</table>

Both investors are endowed with 9 at time 0 and have either 16 or 4 at time 1. In the autarky, each investor has an expected utility of 6. The subjective state prices \( \hat{q}_i = \pi_i u'(c_i)/u'(c_0) \) that measure the trade-offs they cannot make are not equal so they would like to trade. If investor 1 trades 6 units of consumption in state \( a \) for 6 units in state \( b \), then both investors’ expected utilities increase to 6.16. Their subjective state prices are now equal so this is the Pareto optimal solution. The final row shows that each investor could pay up to 0.95 units of consumption to establish this market and still increase his utility. The maximum bearable per capita cost is the solution, \( x \), to \( \mathbb{E}[u] = x u(9 - \text{cost}) + u(10) \).

This is the unique competitive-market solution; however, there are also bargaining solutions that are not based on a competitive equilibrium. For example, if investor 1 has more bargaining power, he might be able to insist on receiving up to 0.95 from investor 2 at time 0 as well. Investor 2 would balk at any higher transfer to investor 1 just as he would to a higher cost of establishing the market.

This problem was simple due to the symmetry and the lack of any aggregate second period risk. If there is risk in the second period a similar result obtains, though it requires more algebra to solve.

To next example illustrates how adding a security can hurt some investors. There are three investors. Investor one has the utility function \( u(c) = c^{1/2} \). Investors two and three have utility functions \( u(c) = -1/c \). There are three equally likely states of nature labeled \( a \), \( b \), \( c \). There is no time-0 consumption. The investors are endowed with time-1 consumption of \( e_1 = (0, 5, 5)' \), \( e_2 = (4, 2, 4)' \), and \( e_3 = (6, 3, 1)' \) in states \( a \), \( b \), \( c \), respectively.

In a complete market, investors with homogeneous beliefs have consumptions that align perfectly across states. This can obviously be true only if each investor has the same consumption in every state. That means that the state prices are equal across states, and as every investor has the same average endowment, their time-0 wealths must be the same. Therefore, in equilibrium, not only is every investor’s consumption across states equal; consumption is also equal across investors. Each consumes 10/3 in all three states. Their expected (and realized) utilities are 1.826, -0.3, and -0.3.

If their endowments are in the form of companies with the given payoffs, each investor arrives at this equilibrium by selling one-third of his company to each of the other two investors in exchange for one third of the other’s company. As the companies are worth the same, this trade is feasible in a price-taking market.

Suppose instead of a complete market, each investor’s endowment is a company with the given payoffs. Shares of companies 1 and 2 trade but company 3 does not trade. This prevents the
investors from achieving a risk-free equilibrium. The equilibrium is described by the prices of companies 1 and 2 and the trades that each investor makes. Because there is no time-0 consumption only the relative price is defined. That is, we can make company 1 the numeraire with a unit price and determine \( p \), the price of company 2. The price of company 3 is undefined as there is it is not traded.

Investor \( k \) purchases the fraction \( \alpha_k \) (or sell if \( \alpha_k < 0 \)) of the second firm. This realizes \( p\alpha_k \) which is used to purchase (or sell) that fraction of the first firm. Investor \( k \)'s consumption in state \( s \) is \( c_{ks} = e_{ks} + \alpha_ke_{2s} - p\alpha_ke_{1s}. \) Each investor solves the problem

\[
\max_{\alpha_k} \sum_s \pi_k u_k(e_k + \alpha_k e_{2s} - p\alpha_k e_{1s}) \Rightarrow \sum_s \pi_k u_k'(e_k + \alpha_k e_{2s} - p\alpha_k e_{2s})(e_{2s} - pe_{1s}) = 0 \quad (47)
\]

These first-order conditions provide three equations for the four unknowns \( \alpha_{1,2,3} \) and \( p \). The fourth equation is the market clearing condition \( \sum \alpha = 0. \) In the resulting equilibrium, investor \( k \)'s subjective state prices are \( \hat{\pi}_k = \pi_k u_k'(c_{ks}). \) The table describes this equilibrium and compares it to the one in the complete market. As only relative prices are defined the subjective state prices have been normalized to sum to 1 in each case.

The market clearing price of company 2 is \( p = 0.936 \) (with the price of company 1 set to 1). In the equilibrium, investor one sells 89.7% of his company to purchase 95.6% of company 2. Investor two purchases 22% of company 1 financed by selling sells 23.5% of his company. Investor three cannot sell his company but purchases 67.7% of company 1 by shorting 72.4% of company 2. The investors have done the best they can with the limited trading opportunities but have not achieved a Pareto optimal allocation of consumption.

<table>
<thead>
<tr>
<th>State</th>
<th>Investor 1</th>
<th>Investor 2</th>
<th>Investor 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Endowment</td>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>Original complete market</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>u</td>
<td>1.826</td>
<td>1.826</td>
<td>1.826</td>
</tr>
<tr>
<td>u'</td>
<td>0.274</td>
<td>0.274</td>
<td>0.274</td>
</tr>
<tr>
<td>( q^k_s )</td>
<td>0.333</td>
<td>0.333</td>
<td>0.333</td>
</tr>
</tbody>
</table>

Investor three would like to consume more in state \( c \), and could afford to give up consumption is state \( b \). But his subjective price for state \( c \) consumption is 0.757, which is high, so it costly to do so. The only way to increase state \( c \) consumption is to buy more of company 1 and short more of company 2. Shorting \( \Delta \alpha \) of company 2 gives up \( 4\Delta \alpha \) of state \( c \) consumption and allows the purchase of 0.936\( \Delta \alpha \) of company 1 generating 5\( \cdot 0.936\Delta \alpha = 4.679\Delta \alpha \) in state \( c \) consumption for a net gain of 0.679\( \Delta \alpha \). However, it also gives up the same amount of state \( a \) consumption with no increase from buying company 1. So the benefit is not sufficient.

The expected utilities of the three investors are

\[
\mathbb{E}[u_1(\tilde{c})] = \frac{1}{3}(1.958 + 1.560 + 2.086) = 1.868 \quad \mathbb{E}[u_3(\tilde{c})] = -0.316 \quad \mathbb{E}[u_3(\tilde{c})] = -0.398. \quad (48)
\]
In the complete market, the investors’ utilities are the same in all states so are equal to the expected utility. Investor one has a higher expected utility in the incomplete market, 1.868, than he does in the complete market, 1.826, so he is hurt by the completion of the market. The reason investor 1 is better off in the incomplete market is that his payoff pattern complements that of investor 3 well. Investor 3 wants to acquire more state $c$ consumption and pay for it with state $a$ consumption. Purchasing company 2 would increase both states $a$ and $c$ consumption. Purchasing company 1 is a much better fit. To do that he must short company 2 as there is no market for his own company. These demands increase the value of company 1 relative to company 2. This increases investor one’s wealth. In the complete market investors one and two have the same wealth. In the incomplete market investor two’s wealth is only 93.6% as much. Investor one’s higher wealth in the incomplete market creates a higher expected utility even though the opportunities to trade are not as good.

Suppose the investors make these trades, and then a market for company 3 opens, completing the market. The state prices will all become 1/3, and now the investors will trade again, starting from new endowment positions. The investors’ initial wealths before this second round of trading are

\[
\begin{align*}
    w_1 &= \frac{1}{3}(3.834 + 2.432 + 4.350) = 3.539 \\
    w_2 &= \frac{1}{3}(3.061 + 2.629 + 4.159) = 3.283 \\
    w_3 &= \frac{1}{3}(3.105 + 4.939 + 1.491) = 3.178.
\end{align*}
\]

These will also be their consumption in each state. Each investor’s consumption is constant across states because completing the market has eliminated aggregate risk. The investors’ utilities will be

\[
\begin{align*}
    u_1 &= \sqrt{3.539} = 1.881 \\
    u_2 &= -1/3.283 = -0.305 \\
    u_3 &= -1/3.178 = -0.315.
\end{align*}
\]

This secondary completion of the market increases the utility of all the investors; however, investor one is better off after the two-step completion than he is if the markets were completed before any trading. The other two investors are better off with a single-step completion. Which progression will occur cannot be modeled without additional assumptions.

### The Representative Investor

To use the pricing relations from this theory there must be some expected utility maximizer for whom we know his (i) beliefs, (ii) portfolio, and (iii) utility function. The usual assumption is a representative investor with (i) correct or consensus beliefs, (ii) holding the market portfolio, and (iii) possessing some mathematically convenient utility function like exponential, CRRA, or EZ. In this context “representative” means specifically average and not just typical. These are all strong assumptions, but under some conditions they can be demonstrated to be true, at least in part.

An obvious first question is: Is there such a representative investor for a given economy? A representative (average) investor always exists in an economy with a complete market when all investors have homogeneous beliefs and state-independent utility. This follows immediately from Theorem 4.1. There it was shown that such investors all hold portfolios which align their time-1 consumption, and that any other portfolio with the same alignment of consumption is also efficient. The latter means that there is some utility function for which the portfolio is optimal.

Because all portfolios with this same ordering of outcomes are optimal, the efficient set is convex when investors have state-independent utility and homogeneous beliefs. Let $e_1$ and $e_2$ be efficient portfolios with outcomes ordered by their indices. Then it is clear that $e = \gamma e_1 + (1-\gamma)e_2$ must also have its outcomes ordered in exactly the same way. This means that the market portfolio, which is a convex combination of efficient portfolios, must be efficient, and, therefore, the representative investor exists.
We have just proved the following theorem which is stated here for future reference.

**Theorem 4.3: Efficiency of the Market Portfolio in a Complete Market.** If markets are complete and all investors are strictly risk averse with state-independent utility and have homogeneous beliefs, then all investors’ optimal portfolios are perfectly aligned and the market portfolio is efficient.

The important condition for this result is not that the market is complete, but that all investors face the same state prices. If this can be established or inferred, then the market will have reached a Pareto-optimal allocation of consumption. This will be true in an effectively complete market.

This theorem shows that making the market effectively complete is a much simpler task than making the market complete. The number of distinct states is huge even if we only need a market of complete prices. Fortunately, not even a market of complete prices is needed for effective completeness.

In an effectively complete market the optimal portfolios for all for investors with state-independent utility and homogeneous beliefs align perfectly, and the efficient set is convex. This means that all optimal portfolios align perfectly with the market portfolio; in any two states in which the market portfolio has the same value, each investors’ optimal portfolios must also have the same value. Therefore, all investors could construct their optimal portfolios from Arrow-Debreu securities created just for each of the aggregated states with distinct market outcomes. Arrow-Debreu securities for every state are not needed. This is a much smaller set, and those securities are easy to create as shown in the next section.

This leaves us only the problem of determining the representative investor’s utility function. Pick any investor \(k\) and order the \(S\) states so that

\[
1 \leq c_{j+1} \leq \ldots \leq c_S.
\]

Per capita consumption is \(c = K^{-1} \sum c_{sk}\). Because all investors' consumptions align, per capita consumption is ordered as in (51) as well. As all investors have homogeneous beliefs, they share a common SDF that is ordered inversely to consumption \(m_1 \geq m_2 \geq \ldots \geq m_S\). The inequalities are strict whenever those in (51) are strict; if consumption is equal in two states then the SDF is as well. Now define the function \(\Upsilon(c) = m_s\). Because \(m_s > 0\), \(\Upsilon\) is an entirely positive function. Because \(c\) and \(m_s\) are inversely ordered, \(\Upsilon\) is a decreasing function. Therefore, \(\Upsilon\) can be interpreted as the marginal utility of the average or representative investor. The integral of \(\Upsilon\) is the utility function of the representative investor.\(^{17}\)

It is equally important to understand what this theorem does not say. By the theorem, the representative investor’s demand for the securities must align with the aggregate demand schedule at the equilibrium prices. However, it need not coincide with aggregate demand at non-equilibrium prices. It does so only when all investors have sufficiently similar utility functions such as LRT utility with the same exponent. This should be obvious. The pricing restrictions only impose \(S\) conditions on the utility function fixing its derivative at \(S\) points. Many positive monotonic functions can be fit to those points, and they will generate different demands if prices are changed.

**Creating the Necessary Derivative Assets**

\(^{17}\) In a discrete state space, this integration is a sum and gives the utility function only at a set of discrete points. The utility function is not unique. The functional form must be picked so that it is differentiable at those points. This is seldom a concern as it is the marginal utility function that is always used in pricing. See Constantinides (1982) for a complete development.
Denote the value of the market portfolio in aggregated state \( a \) by \( M_a \), and order the states from lowest to highest market outcome, \( M_1 < M_2 < \ldots \). To create a pure state security for aggregate state \( a \), consider three call options with strikes \( M_{a-1} \), \( M_a \) and \( M_{a+1} \). Their payoffs in state \( s \) with a market value of \( M_s \) are

\[
C_{a-1}(s) = \max(M_s - M_{a-1}, 0) \quad C_a(s) = \max(M_s - M_a, 0) \quad C_{a+1}(s) = \max(M_s - M_{a+1}, 0)
\]  

(52)

A call spread long the second call and short the first will be worth zero in states \( a - 1 \) and below and worth the constant amount \( M_a - M_{a-1} \) in states \( a \) and above. Similarly a call spread long the third and short the second will be worth zero in states \( a \) and below and worth the constant amount \( M_{a+1} - M_a \) in states \( a + 1 \) and above. Therefore, a portfolio long \((M_a - M_{a-1})^{-1}\) of the first spreads and short \((M_{a+1} - M_a)^{-1}\) of the second spreads will be worth 1 in state \( a \) and 0 in all other states. This is the desired pure state security; its value is the state price \( q_a \).

This construction involves a second difference in strike prices. For a continuous state space, we can achieve the same result with a second derivative with respect to the strike price. We can now simply denote the state by the value of the market portfolio, \( M \). The value of an option with a strike of \( K \) is

\[
C(K) = \int_0^\infty \max(M - K, 0)Q(M)dM = \int_K^\infty (M - K)Q(M)dM.
\]  

(53)

The second derivative of this price with respect to its strike price is

\[
\frac{\partial^2 C(K)}{\partial K^2} = \frac{\partial}{\partial K} \left[ \int_K^\infty (M - K)Q(M)dM \right] = \frac{\partial}{\partial K} \left[ \int_K^\infty Q(M)dM \right] = Q(K).
\]  

(54)

With this construction, we have eliminated the need to know the representative investor’s utility function and probability beliefs. All such information is embedded in the option prices and can be extracted so that the state price of every aggregate state with the same market outcome can be determined.

Valuation with these aggregate-state prices involves one additional step because a given aggregated state does not necessarily specify the payoff on every asset as the original states do. For each state, the state price is proportional to the states probability and the marginal utility of consumption in that state. Within each aggregate state (denoted by \( a \)), consumption, and, therefore, the marginal utility of consumption is the same

\[
q_s = \pi_s \frac{\partial U(c_0, c_1)}{\partial c_1} = \pi_s \frac{\partial U(c_0, c_1)}{\partial c_0} = \pi_a \frac{\partial U(c_0, c_1)}{\partial c_1} \equiv \pi_a \delta a Q_a
\]  

(55)

That is, the state price for state \( s \) is equal to its aggregated state price multiplied by the probability that state \( s \) occurs conditional on its aggregate-state occurring. The price of asset \( i \) is then

\[
p_i = \sum_a Q_a \sum_{s:a} \pi_{i,a} X_{si} = \sum_a Q_a \mathbb{E}[\hat{X}_i | a].
\]  

(56)

So to value any asset we need to know its conditional expected payoff in each market aggregate-
state as well as the aggregate-state prices. Here, of course, it is important that beliefs are homogeneous so this expectation is the common one.

The theorem proved previously remains valid. If investors have homogeneous beliefs and there is a complete set of state securities for every aggregate-state with a distinct outcome for the market portfolio, then the market portfolio is efficient, and a representative investor exists.