

## An Introduction to the Language of this Class.

Definition: A *project* is anything that generates a series of cash flows.

Examples:

A. You start a bicycle repair shop. Cost \$500. Expected earnings \$1,000 per month starting next month for 24 months.

B. A company decides to sell purple pencil lead and expects to earn \$10,000 per year for 10 years beginning next year after an initial investment of \$20,000.

C. An investor purchases a Certificate of Deposit (CD) from a bank for \$10,000. The bank will repay \$12,000 in two years.

D. A lottery ticket that costs \$1, and offers to pay \$14,000,000 with some probability.

Projects can range from true physical investments (the bicycle repair shop), to pure monetary investments (the CD), to gambles (the lottery ticket).

*Whatever the source of the funds, finance only concerns itself with the actual cash flows.*

## **A Firm.**

A firm or company invests in various projects.

These projects generate expenses, and produce revenues.

Revenues in excess of expenses go to either new investments, or to the firm's security holders.

# **An Overview of Corporate Securities.**

## **Common stock (equity).**

*Cash Flows:* Receives *all* corporate payouts, after every other claimant has been paid. Payments to the equity holders are called *dividends*.

Larger firms have their common stock traded on a major *stock exchange*, like the New York Stock Exchange. There investors can purchase and sell shares. Closing prices for the previous days trade can be found in most newspapers.

*Tax Status:*

**Individuals.**

*Dividends* are taxed as “ordinary income” (just like wage income).

*Capital Gains* have historically been taxed at a rate below that of ordinary income. (A shareholder who purchases a share of stock for \$20, and sells it for \$22, realizes a \$2 capital gain.)

**Company.**

Payments to equity are not tax deductible from corporate profits.

*Corporate Rights:*

Can hire and fire management, its voting power allows its owners to control corporate decision making.

# Preferred stock.

## *Cash Flows:*

The firm agrees to give the holder of this security a set dividend every period (for example \$2.25 per calendar quarter per share).

1. Like common stock, preferred stock may be traded on a stock exchange. If it is traded on an exchange, the newspaper will report the closing price.

### *Tax Status:*

Same as common stock.

1. Very poor tax treatment. Payments resemble debt, but are not tax deductible by the firm. Tax consequences to the individual are the same as with debt.
2. Due to tax treatment, preferred stock is only issued by regulated firms (i.e. public utilities issue this security).

### *Corporate Rights:*

Paid before common equity, but after all other claimants. Generally, cannot vote. If the company fails to make a payment preferred stockholders cannot force any immediate action.

Contracts generally prevent common stockholders from receiving funds until missed payments are made up, often with interest.

# Corporate Debt (Bonds).

*Cash Flows*: First in line for payment.

Contracts require a fixed *coupon* payment every 6 months. At maturity the firm then pays \$1000 which equals the *face value* of the bond.

Larger firms have their bonds traded on a major *bond exchange*, like the New York Bond Exchange. There investors can purchase and sell bonds. Closing prices for the previous days trade can be found in most newspapers. However, most bond transactions take place between brokers rather than on an exchange.

*Tax Status:*

### **Individuals.**

*Coupon payments* are taxed as “ordinary income” (just like wage income).

*Capital Gains:* The difference between what an investor pays for the bond and its price when sold. If the investor holds the bond to maturity then the difference between the purchase price and the face amount is considered a capital gain.

### **Company.**

Payments to debt *are* tax deductible from corporate profits.

*Corporate Rights:*

If the firm misses a payment the bondholders have the right to force a bankruptcy proceeding. In principle this allows them to take over the company.

# Types of Debt

## I. *Maturity.*

1. Funded, any debt repayable in more than one year.
2. Unfunded, debt repayable in less than one year.

## II. *Repayment provisions.*

1. Sinking fund. The company contributes money to the fund which then repurchases the bonds.
2. Call options give the firm the right to repurchase the debt at a specified price. (Usually, cannot do this for at least 5 years.)

## III. *Seniority.*

- A. Senior Debt: Paid prior to all other claimants.
- B. Subordinated Debt: Paid after the senior debt holders.
- C. Secured Debt: Can claim the asset used as security if payments are not satisfied. *A lease* agreement is essentially the same as secured debt. The difference lies in which party technically owns the asset. With secured debt the firm owns the asset, in a lease the bank owns the asset.

#### IV. *Rates.*

A. Floating rate: the interest rate is tied to some variable in the economy. For example adjustable rate mortgages may be set 2.5 points above the one year treasury bill rate.

B. Fixed rate: the interest rate is fixed throughout the life of the loan.

#### V. *Default risk.*

A. "Junk bonds": A low rated bond, with a high probability that the firm will not meet its contractual repayment obligations.

B. Investment grade: A highly rated bond, low probability of default.

## VI. *Hybrid Security.*

Convertible bonds can be exchanged for stock at a specified price, at the lender's option.

## VII. *Placement.*

A. Publicly placed debt is sold to the general public and traded in the open market.

B. Privately placed debt is purchased by large institutions and either not traded at all or traded only among large institutions.

# Derivatives

Definition: *A derivative is any security whose value derives from another security.*

There now exist a wide array of derivative securities. Below are a few of the more common types.

## *Call Options.*

American: gives the holder the right to purchase one share at a fixed price at any time on or prior to some date.

European: gives the holder the right to purchase one share at a fixed price at some date.

## **Terminology:**

*Strike Price:* the price at which the option holder can purchase the stock.

*Expiration Date:* the final date at which the option can be used.

*Exercising an Option:* to use the option to purchase stock.

*Warrant:* A call option issued by a firm to either its employees or investors.

### *Put Options.*

Same as a call option except that it gives the owner the right to *sell* one share at a fixed price.

On expiration how much is a **call** option worth?

Case 1: The strike price is *greater* than the price of the stock.

In this case the investor can obtain a share of the stock for less money by purchasing it in the open market. So the option should *not* be exercised.

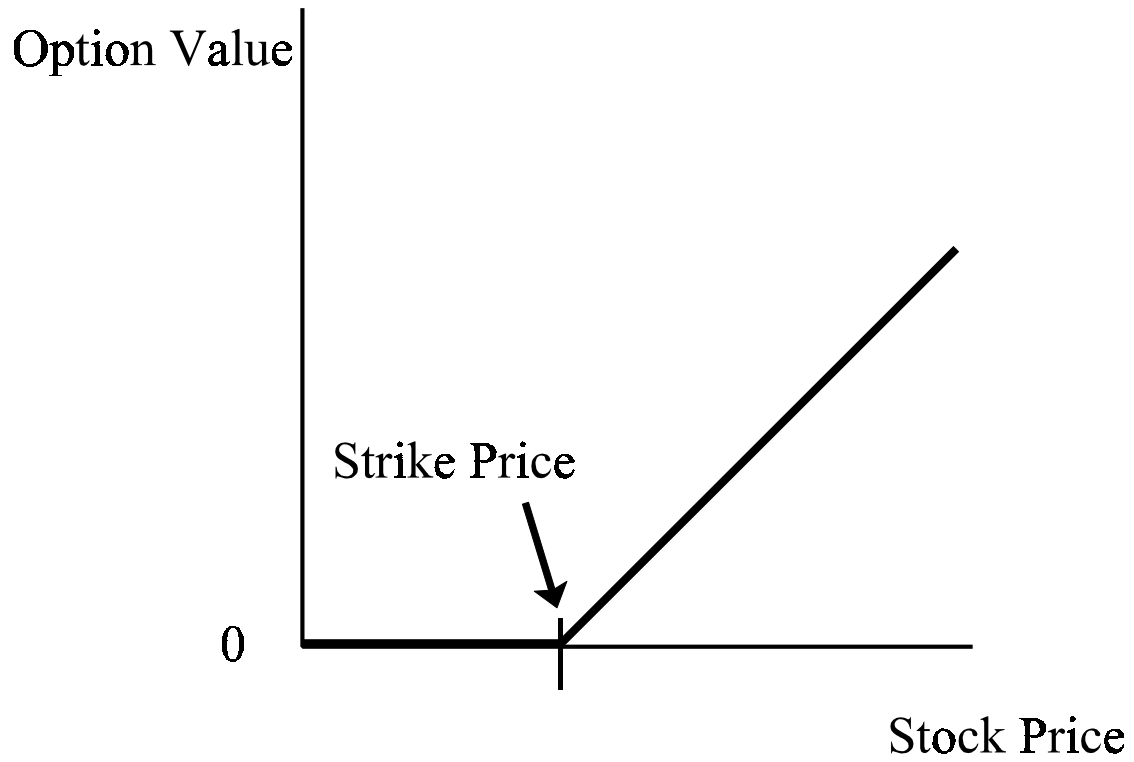
Option's value = 0.

Case 2: The strike price is *less* than the price of the stock.

In this case exercising the option is less expensive than purchasing it in the open market. So the option *should* be exercised.

Option's value =  
Market Price of the Stock - Exercise Price.

# The Value of a Call Option at Expiration



On expiration how much is a **put** option worth?

Case 1: The strike price is *greater* than the price of the stock.

In this case the investor can sell a share of the stock for more money by using the option. So the option *should* be exercised.

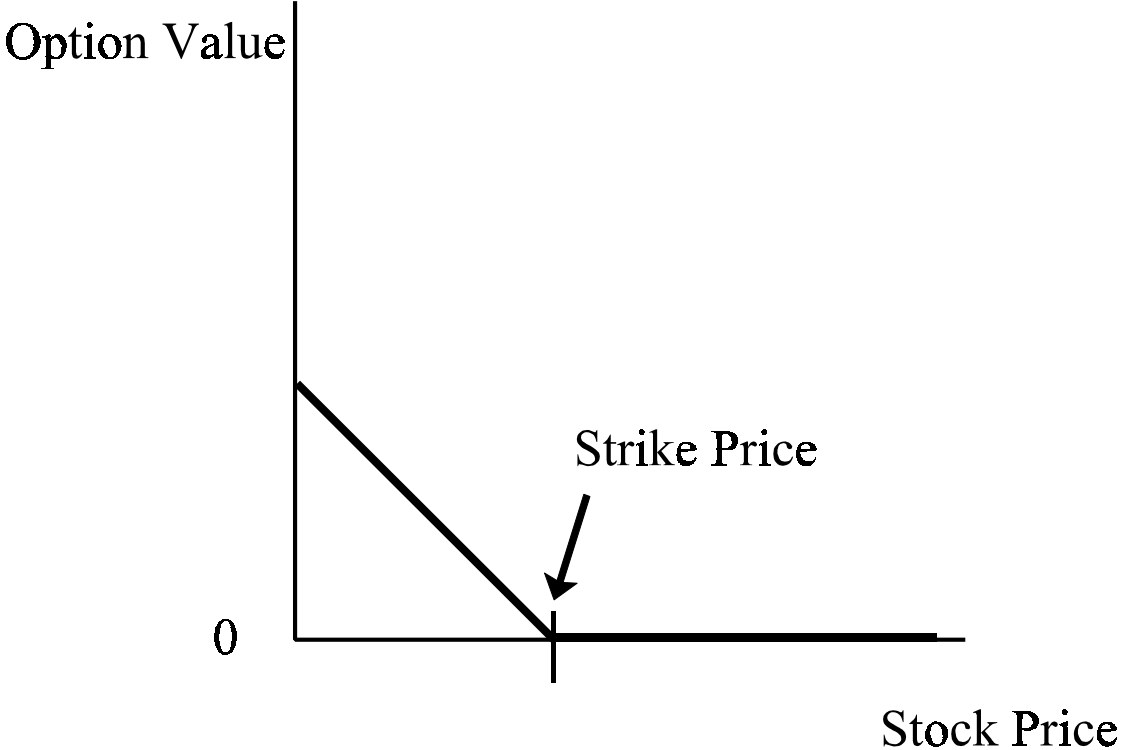
Option's value =  
Exercise Price - Market Price of the Stock.

Case 2: The strike price is *less* than the price of the stock.

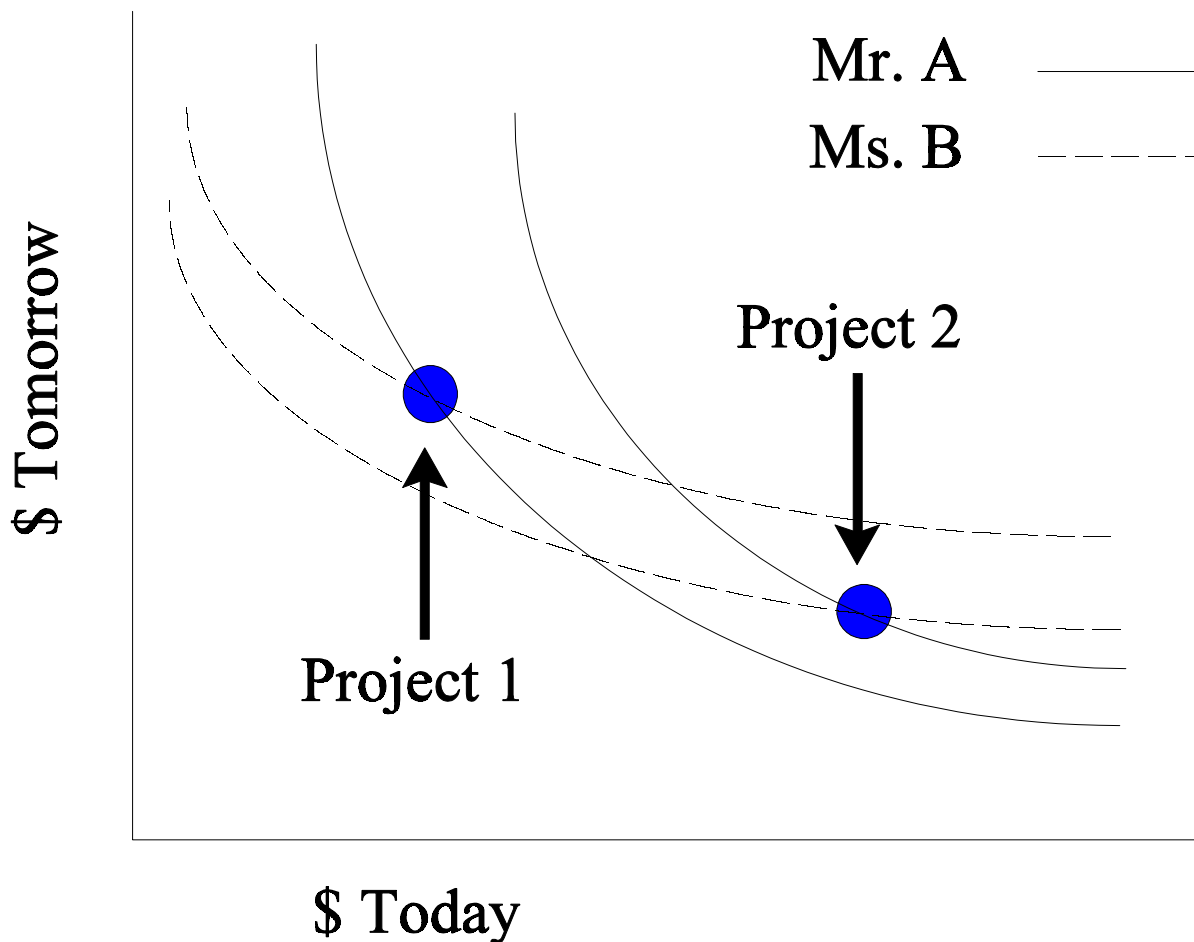
In this case the investor can sell the stock for more in the open market. So the option should *not* be exercised.

Option's value = 0.

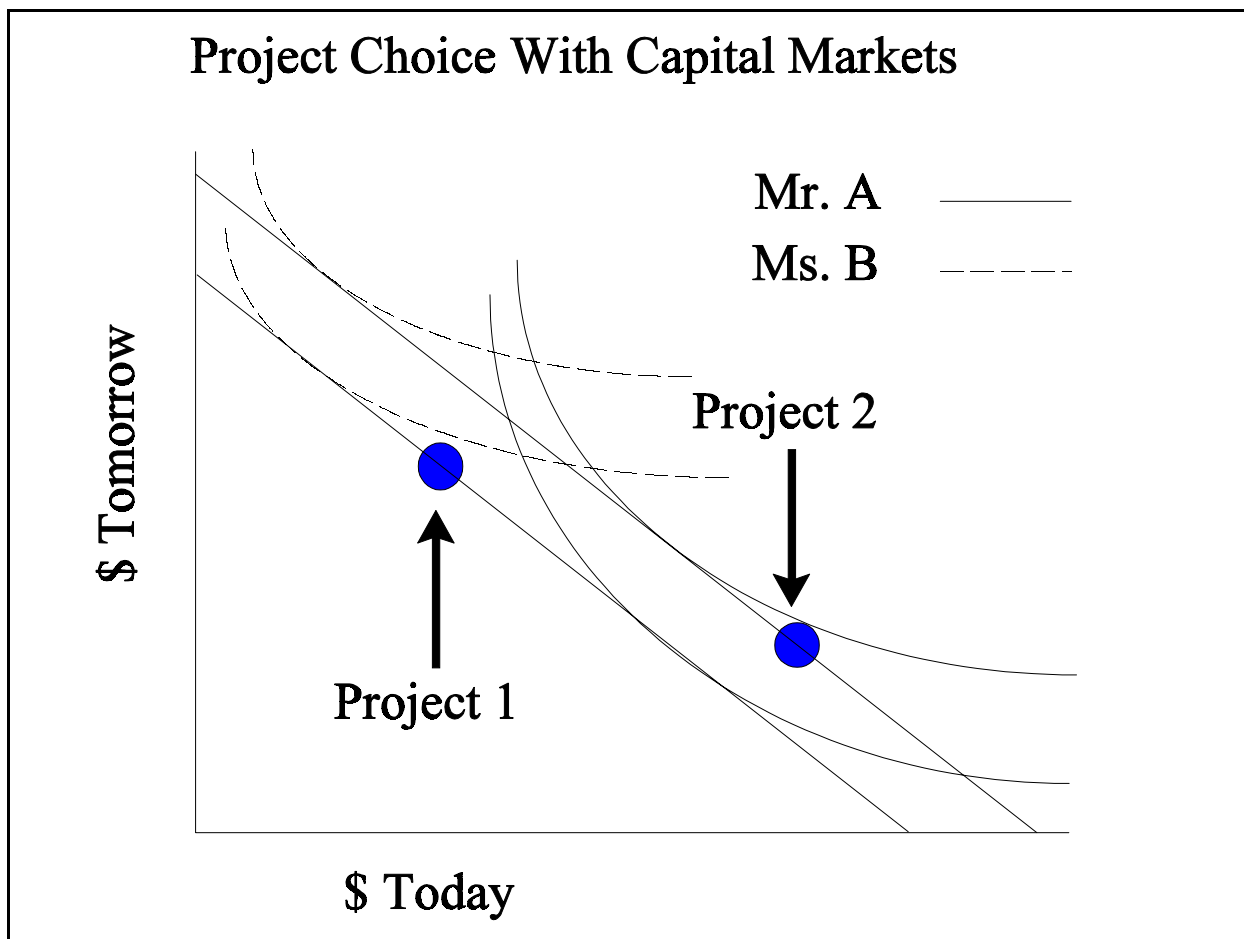
# The Value of a Put Option at Expiration



## Project Choice Without Capital Markets



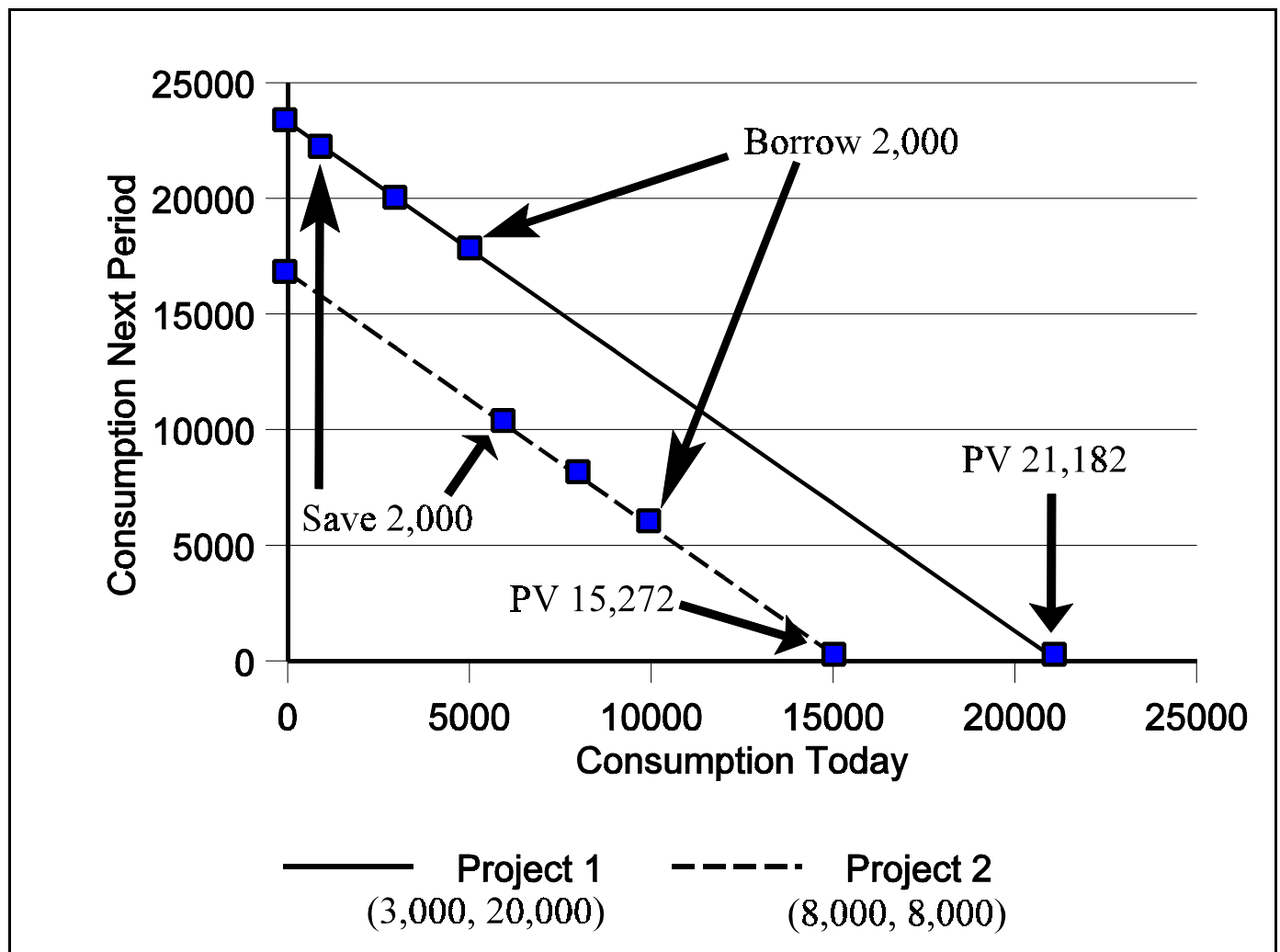
Here we see Mr. A prefers project 2, while Ms. B prefers project 1. Without knowing an investor's consumption preferences, and absent a capital market, it is impossible to say one project is superior to another.



Both investors prefer project 2 to project 1. Note, the budget constraint induced by project 2 intersects the \$ Today axis further to the right. The further to the right the intersection the higher the investor's welfare. The point of intersection is called the Present Value of the project.

With a capital market all investors agree on the ranking of projects. The project with the highest Present Value (PV) is best. Thus, if an investment advisor only knows each project's PV she can rank them for her clients.

## Consumption Shifting Example. Interest rate 10%.



Consume All Today & Zero Next Period

$$PV(\text{Project 1}) = 3,000 + 20,000/1.1 = 21,182$$

$$PV(\text{Project 2}) = 8,000 + 8,000/1.1 = 15,272$$

## Borrow 2,000

Project 1: Consume 5,000 today and  $20,000 - 2,000 \times 1.1 = 17,800$  next period.

Project 2: Consume 10,000 today and  $8,000 - 2,000 \times 1.1 = 5,800$  next period.

## Save 2,000

Project 1: Consume 1,000 today and  $20,000 + 2,000 \times 1.1 = 22,200$  next period.

Project 2: Consume 6,000 today and  $8,000 + 2,000 \times 1.1 = 10,200$  next period.

## Save Everything

Project 1: Consume 0 today and  $3,000 \times 1.1 + 20,000 = 23,300$  next period. The figure 23,300 is called the Future Value of the project.

Project 2: Consume 0 today and  $8,000 \times 1.1 + 8,000 = 16,800$  next period. The figure 16,800 is called the Future Value of the project.

## The Investor's Problem: An example.

Suppose an investor owns project 1 and has a utility function over consumption equal to  $U=X_1X_2$ , where  $X_1$  and  $X_2$  equal period 1 and 2 consumption levels.

How much will the individual consume in each period?

Goal: Maximize  $U$ .

Constraint:

$$X_2 = (3,000 - X_1)(1.1) + 20,000.$$

Savings from period  
1, plus interest.

Period 2 cash  
flow.

Finding the solution: Use the constraint to eliminate  $X_2$  from the utility function, and then maximize with respect to  $X_1$ .

Maximize  $U = X_1 \{ (3000 - X_1)(1.1) + 20000 \}$  with respect to  $X_1$ .

Differentiate with respect to  $X_1$ , and set the result equal to zero

$$(3000 - X_1)(1.1) + 20000 - 1.1X_1 = 0.$$

Now solve for  $X_1$ :

$$X_1 = 10,591.$$

Given  $X_1$ , the constraint now tells us what  $X_2$  equals,

$$X_2 = (3000 - 10591)(1.1) + 20000 = 11,650.$$

Notice that the investor maximizes his utility by *borrowing* 7,591, to finance period one consumption.

What happens if the investor only has project 2 available? In this case the constraint becomes

$$X_2 = (8,000 - X_1)(1.1) + 8,000.$$

Repeating the analysis shows that the optimal choices for  $X_1$  and  $X_2$  equal:

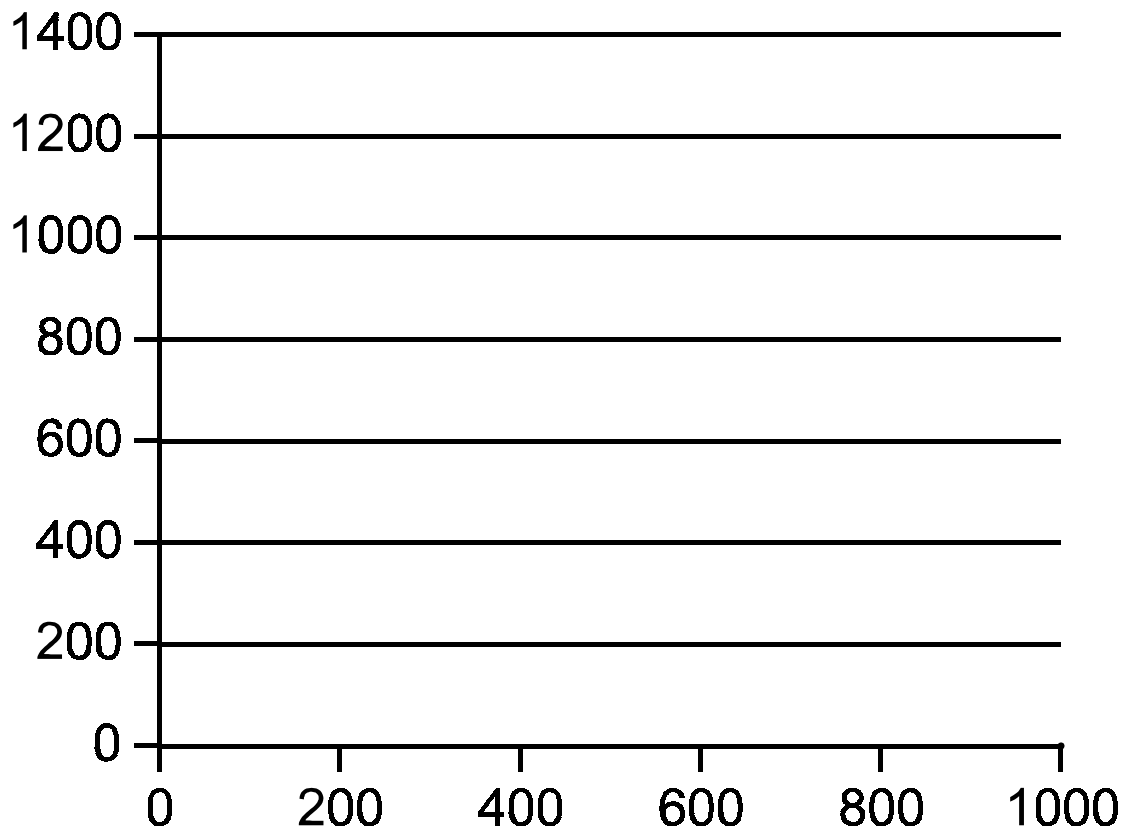
$$X_1 = 7,636, \text{ and } X_2 = 8,400.$$

Since project 2 induces lower consumption levels in both periods the investor is worse off.

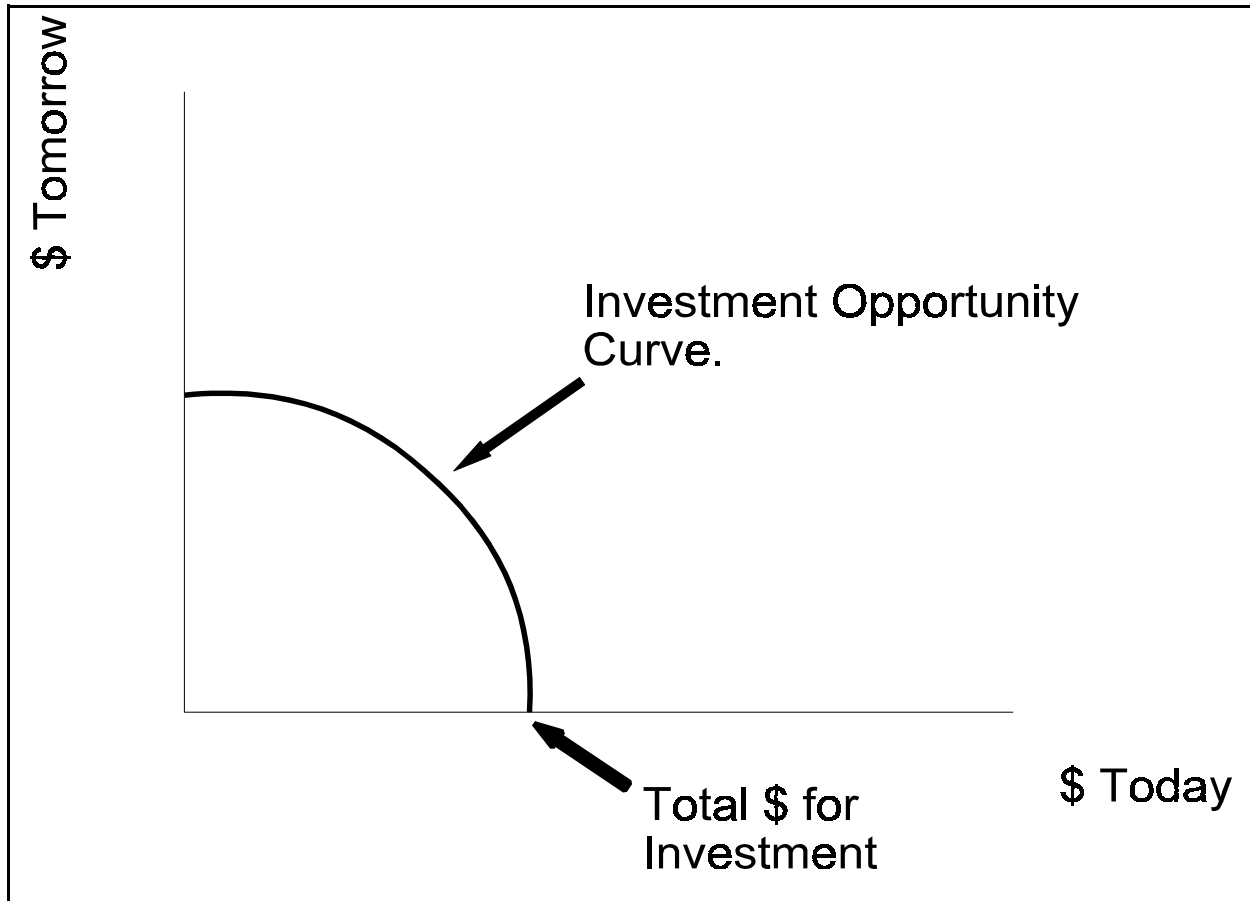
Choosing Among Many Projects: You have \$1,000 in the bank, and a corn farm. Each row of corn costs \$200 to seed, and there are 5 rows. The payoff for each row is listed below:

Row: A 50, B 600, C 400, D 200, E 100

Plot the tradeoff between money today, and money tomorrow.

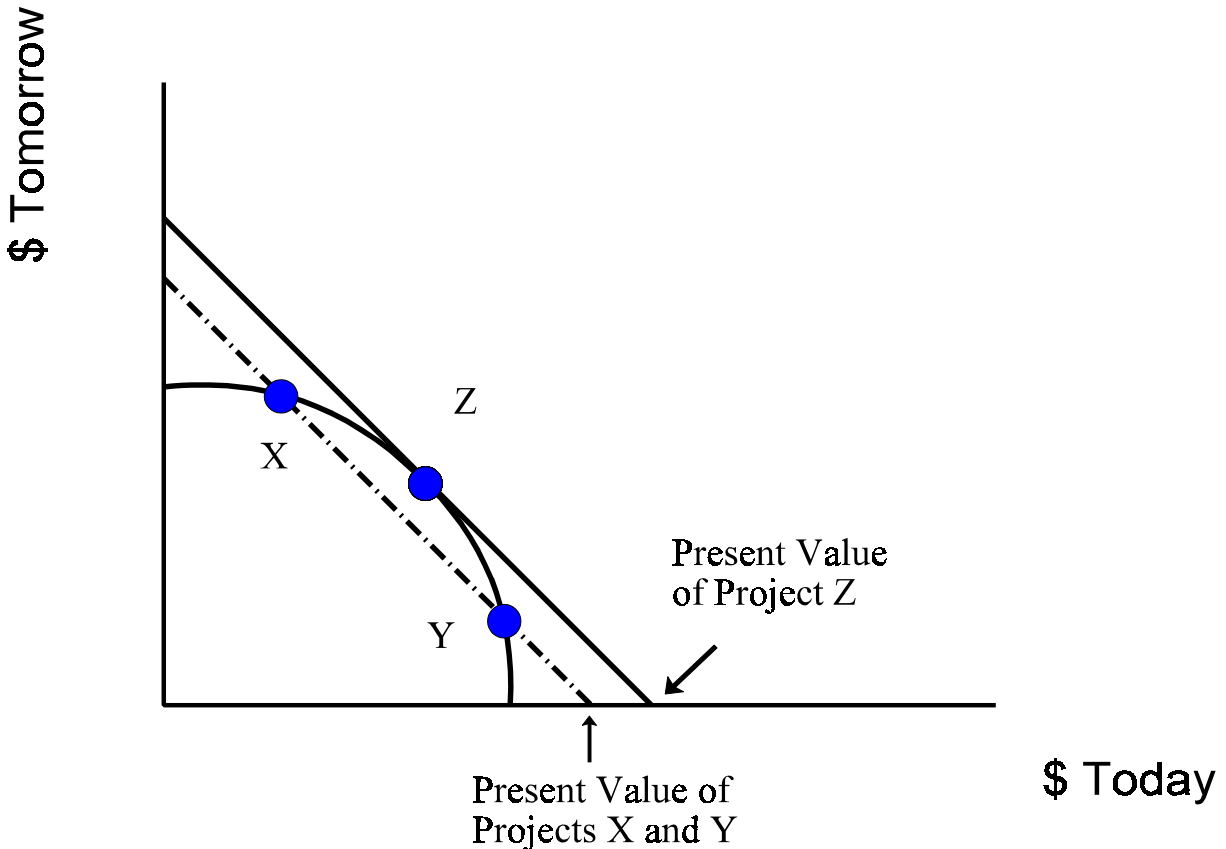


In general the investment opportunity curve has the following shape.

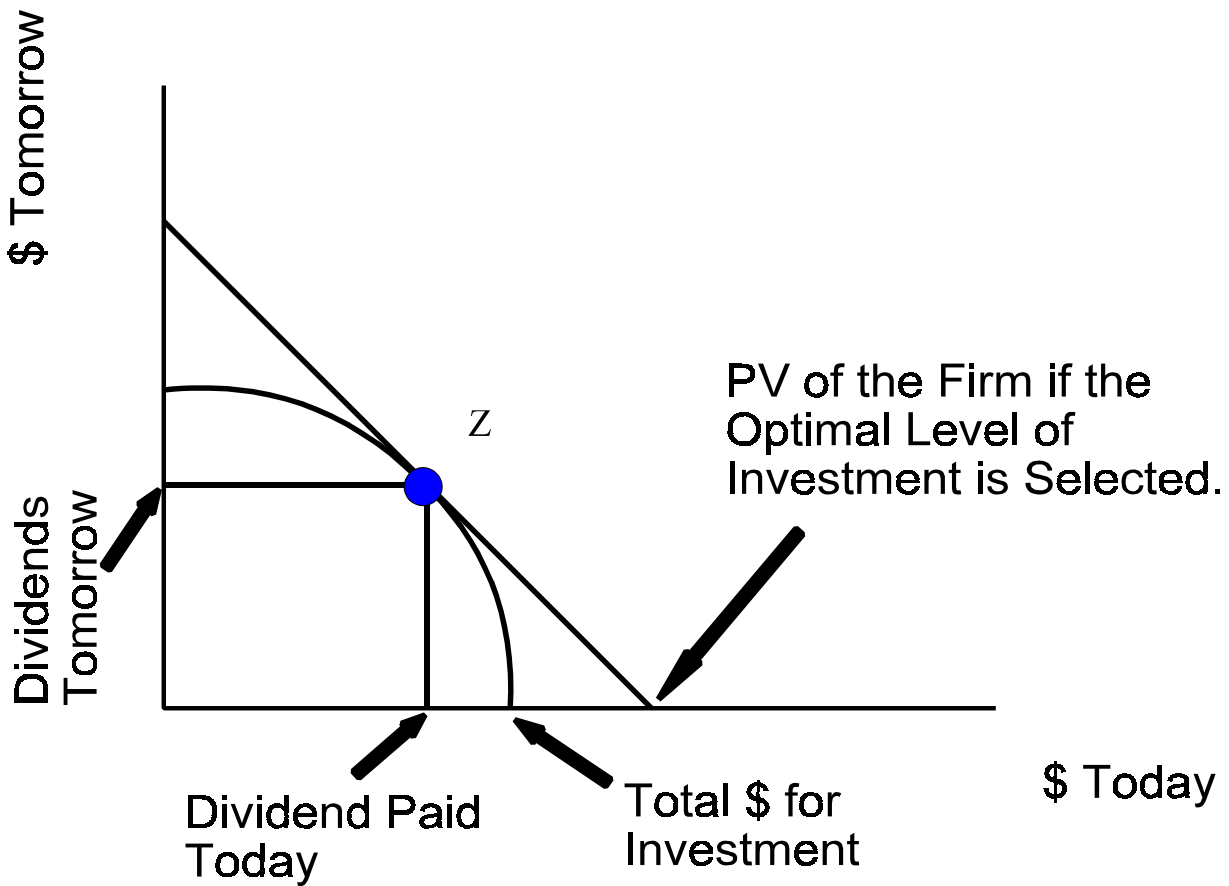


The investment opportunity curve reflects the firm's ability to trade off dividends today (via forgone investment) for dividends tomorrow.

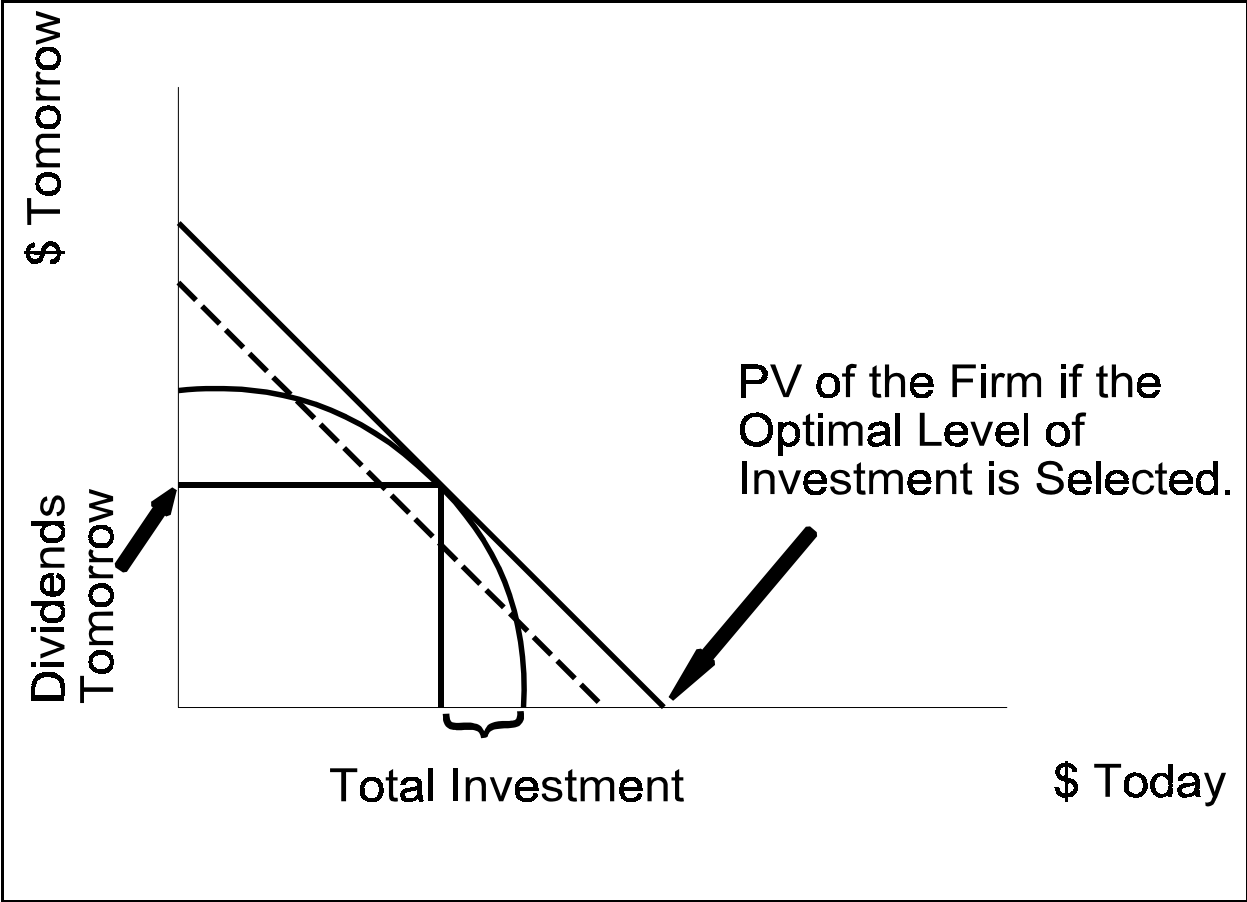
Why is the curve concave? Start all the way to the right. The first dollar invested goes into the best project. Suppose it returns \$2. The second dollar goes into the second best project, returning \$1.99. The third goes into the third best project returning only \$1.98, etc. Continuing on, the curve gets flatter and flatter as you go to the left.



Displayed are three projects on the investment opportunity curve. Selecting X or Y allows the investor to consume along the dashed line by accessing the capital markets. If project Z is selected then consumption can take place along the solid line. Project Z is the optimal project. The principle that investors want the project with the highest PV continues to hold.



The difference between the "Total \$ for Investment" and the "Dividend Paid Today" points, equals the amount invested. Which is displayed on the next graph.



The optimal investment level occurs when the slope of investment opportunity curve equals  $-(1+r)$ , where  $r$  is the interest rate.

Notice that the PV of the firm under an optimal investment policy exceeds the amount of money it has available to invest today.

## What is Present Value?

Given a set of cash flow  $C_0, C_1, C_2, \dots$  PV is the amount you can have today if you borrow fully against all the cash flows. Thus, the bank gives you PV today, you agree to turn over the cash flows  $C_0, C_1, C_2, \dots$

Let  $C_t$  represent a cash flow arriving in year  $t$  and  $r$  the interest rate. Then

$$PV = C_0 + \frac{C_1}{1+r} + \frac{C_2}{(1+r)^2} + \frac{C_3}{(1+r)^3} + \dots$$

Example:  $C_0=2$ ,  $C_1=7$ ,  $C_2=8$  and  $r=.09$

$$PV = 2 + \frac{7}{1.09} + \frac{8}{(1.09)^2} = 15.16$$

How to calculate PVs with the HP-12C

Step 1: Clear all registers.

Step 2: Enter data.

{2}[g][CF <sub>0</sub> ]	(Period 0 cash flow)
{7}[g][CF <sub>j</sub> ]	(Period 1 cash flow)
{8}[g][CF <sub>j</sub> ]	(Period 2 cash flow)
{9}[i]	(Interest rate)
[f][NPV]	(Answer)

Second example:  $C_0=-90$ ,  $C_1=150$ ,  $C_2=0$ ,  
 $C_3=275$  and  $r=.09$

$$\begin{aligned} PV &= -90 + \frac{150}{1.09} + \frac{0}{(1.09)^2} + \frac{275}{(1.09)^3} \\ &= 259.97 \end{aligned}$$

How to calculate this with the HP-12C

{90}[CHS][g][CF<sub>0</sub>]

{150}[g][CF<sub>j</sub>]

{0} [g][CF<sub>j</sub>]

{275}[g][CF<sub>j</sub>]

{9}[i]

[f][NPV] = answer

Third example: The Lottery.

You know that the odds of winning the lottery is 13 million to 1. Walking by the news stand you see a big sign "JACKPOT 20 MILLION!!<sup>†</sup>"  
Should you invest \$1 for a ticket?

Looking on the back of the ticket you notice the footnote. Then remembering what you learned in your finance course you begin calculating the present value of the payoff. A quick examination of the *Wall Street Journal* shows that the appropriate interest rate is 8 percent. Next you recall, that unfortunately when you win so does the federal government (ie. the winnings are taxable). So you can only expect to keep 70 percent of the payoff each year.

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<sup>†</sup>Paid in 20 annual equal installments.

In net here is the problem, you have one chance in 13 million of winning a 20 year cash flow equal to  $(\text{After Tax Rate})(\text{Payoff})/(\text{Years}) = .7(20,000,000)/20=700,000$ . Plugging this into the HP-12C yields: 6,872,703. (Assuming of course you are the only winner, otherwise you split the jackpot. With odds like these you are literally better off putting your money on #16 at a Reno roulette wheel.)

Entering the information in the HP-12C:

{20}[n]	20 years.
{8}[i]	8 percent interest rate.
{700000}[PMT]	Annual payment.
{0}[FV]	No future value.
[PV]	Get the answer.

The calculator gives a negative answer since it views the PMT, as a cash outflow. Enter {700000}[CHS][PMT] to get a positive answer.

What is an interest rate?

Definition: Suppose  $r_n$  represents an  $n$  period interest rate. Then an agent investing  $C$  dollars, will have at the end of  $n$  periods  $C(1+r_n)$  dollars.

*Only* numbers that meet the above definition are interest rates.

Compounding: How to go from short periods to long periods.

Example A: The 1 month interest rate is 1%.  
What is the 1 year rate?

There are 12 months in 1 year.

$$(1.01)^{12} = (1 + r_{\text{yearly}})$$

Interest received  
after 12 months  
in the bank.

Interest received  
after 1 year  
in the bank.

$$r_{\text{yearly}} = .1268$$

How to calculate this with the HP-12C

{1.01}[ENTER]  
{12}[y<sup>x</sup>]

Places 1.01 in memory.  
Takes 1.01 to the 12th.

Example B: The 1 day rate is .2%. What is the weekly rate?

There are 7 days in a week.

$$(1.002)^7 = (1+r_{\text{weekly}})$$

Interest received  
after 7 days  
in the bank.

Interest received  
after 1 week  
in the bank.

$$r_{\text{weekly}} = .014$$

How to go from long periods to short periods.

Example A: The annual interest rate is 14%.  
What is the daily rate?

In 1 year there are 365 days.

$$1.14 = (1+r_{\text{daily}})^{365}$$

Interest received  
after 1 year  
in the bank.

Interest received  
after 365 days  
in the bank.

$$r_{\text{daily}} = .000359$$

How to calculate this with the HP-12C

{1.14}[ENTER]  
{365}[1/x][y<sup>x</sup>]

Example B: The monthly (month of February) interest rate is 1.5%. What is the weekly rate?

In 1 month there are 4 weeks.

$$1.015 = (1+r_{\text{weekly}})^4$$

Interest received  
after 1 month  
in the bank.

Interest received  
after 4 weeks  
in the bank.

$$r_{\text{weekly}} = .003729$$

What they say at banks.

Sign in the window reads:

Interest rate: 8% compounded daily.

Effective annual yield: 8.33%.

How the bank calculates your daily balance:

Each day they multiply your balance by

$$1 + .08/365$$

Thus, over 1 year you earn

$$\left[ 1 + \frac{.08}{365} \right]^{365}$$

times your initial investment.

Question: Is 8% an annual interest rate?

Answer: NO! If 8% was an annual interest rate, then placing \$100 in the bank for 1 year should produce  $100(1.08) = 108$  dollars.

Instead, you earn

$$100 \left[ 1 + \frac{.08}{365} \right]^{365} = 108.33$$

Question: Is the "effective annual yield" an interest rate?

Answer: YES!

Question: What is the daily interest rate?

Answer:  $.08/365$

What they say at six month treasury auctions.

At a treasury auction, they sell bills that pay \$10,000 in 180 days. If the quoted discount rate is 10%, then it means you can purchase the bill at the auction for \$9,500.

Price of the bill =  
 $100(100 - (\text{days to maturity}/360)\text{discount rate})$ .

$$9,500 = 100(100 - (180/360)10)$$

Note t-bills are quoted in units of 100. So the actual quote will be

$$95 = 100 - (180/360)10.$$

Question: Is 10% an true interest rate?

Answer: No. If 10% is a true annual interest rate then placing \$10,000 into t-bills for one year should leave you with \$11,000. In fact you earn  $(10,000/9,500)^2 10,000 = \$11,080$ .

The actual 6 month interest rate is (ignoring the fact that there are 365 and not 360 days in a year):

$$9,500(1+r_{6 \text{ month}}) = 10,000$$

$$r_{6 \text{ month}} = .0526.$$

So the annual interest rate equals,

$$1 + r_{\text{annual}} = (1+r_{6 \text{ month}})^2 = 1.108.$$

This means the true interest rate is slightly higher than the quoted discount rate.

An example drawn from an actual automobile loan agreement.

The advertisement says 12 month car loans.  
Only 9%!

How they calculate the payments on a \$10,000 car.

A 12 month \$10,000 loan, at 9% implies that you owe \$10,900. Twelve equal payments come out to  $10,900/12 = 908.33$  per month.

What is the actual interest rate on the loan?

The true interest rate must set the present value of the payments equal to the initial loan.

$$10,000 = \frac{908.33}{1 + r_{\text{monthly}}} + \frac{908.33}{(1 + r_{\text{monthly}})^2} + \dots + \frac{908.33}{(1 + r_{\text{monthly}})^{12}}$$

Therefore  $r_{\text{monthly}} = 1.35\%$ , and the annual interest rate  $r_{\text{annual}} = (1 + r_{\text{monthly}})^{12} - 1 = 17.5\%$  percent per year!

How to find these numbers on your calculator.

{10000}[CHS][g][CF <sub>0</sub> ]	Loan amount.
{908.33}[g][CF <sub>j</sub> ]	Payments.
{12}[g][N <sub>j</sub> ]	No. of payments.
[f][IRR]	Calc. 1 mo. rate.

(Note that the 1.35 is a percentage)

Why is the actual interest rate so high?

Each month you pay off part of the principal, thereby borrowing less later in the year. Yet the interest rate calculated by the dealer assumes that you borrow all \$10,000 for the whole year.

A true loan with a 9% annual interest rate comes out to 12 monthly payments of \$872.89.

How to find the monthly payments. If the annual interest rate equals 9%, then monthly rate equals  $(1.09)^{1/12} = 1+r_{\text{monthly}}$ ,  $r_{\text{monthly}} = .0072073$ . The monthly payment therefore solves:

$$10,000 = \frac{\text{PMT}}{1.0072} + \frac{\text{PMT}}{(1.0072)^2} + \dots + \frac{\text{PMT}}{(1.0072)^{12}}$$

Using your calculator:

{10000}[CHS][PV]	Loan.
{.72073}[i]	Monthly rate.
{12}[n]	No. of payments.
{0}[FV]	No balloon payment.
[PMT]	Calc. mo. payment.

Topic: Discounting when the interest rate changes over time.

Let  $f_t$  represent the interest rate in period  $t$ . This is called the year  $t$  forward rate. (Note: the text book talks about spot rates,  $f_t$  as defined here is a forward rate.)

Question: How much does a person earn after leaving  $C$  in the bank for 1 year?

Answer:  $C(1+f_1)$

Question: How much does a person earn after leaving  $C$  in the bank for 2 years?

Answer:  $C(1+f_1)(1+f_2)$

Earnings after 1 year.  $\times$  Interest rate on the 2nd year

Question: How much does a person earn after

leaving  $C$  in the bank for 3 years?

Answer:  $C(1+f_1)(1+f_2)(1+f_3)$

Earnings after 2 years.  $\times$  Interest rate on the 3rd year

Based upon what a person can earn by placing his money in a bank one can conclude that the PV formula is:

$$PV = C_0 + \frac{C_1}{(1+f_1)} + \frac{C_2}{(1+f_1)(1+f_2)} + \frac{C_3}{(1+f_1)(1+f_2)(1+f_3)} + \dots$$

Example:  $C_0=10, C_1=11, C_2=12, f_1=.14, f_2=.15$

$$PV = 10 + \frac{11}{1.14} + \frac{12}{(1.14)(1.15)} = 28.8$$

## Spot rates vs. forward rates:

The text gives the following PV formula when the interest rate varies over time. ( $r_t$  = the year  $t$  spot rate.)

$$PV = C_0 + \frac{C_1}{1 + r_1} + \frac{C_2}{(1 + r_2)^2} + \frac{C_3}{(1 + r_3)^3} + \dots$$

What is a spot rate?

A  $t$  year spot rate is the rate you earn on an annual basis when you leave your money in the bank for  $t$  years.

How do spot rates relate to forward rates?

$$1+f_1 = 1+r_1$$

$$(1+f_1)(1+f_2) = (1+r_2)^2$$

$$(1+f_1)(1+f_2)(1+f_3) = (1+r_3)^3$$

So the  $t$  year spot rate is the geometric mean of the year 1 through  $t$  forward rates.

## Examples:

1. Using forward rates to calculate spot rates.

If  $f_1=.14$ ,  $f_2=.15$ ,  $f_3=.10$ , then

$$r_1 = .14$$

$$(1 + r_2)^2 = (1.14)(1.15)$$

$$r_2 = \sqrt{(1.14)(1.15)} - 1 = .1450$$

$$(1 + r_3)^3 = (1.14)(1.15)(1.10)$$

$$r_3 = [(1.14)(1.15)(1.10)]^{1/3} - 1 = .1298$$

2. Using spot rates to calculate forward rates.

If  $r_1=.05$ ,  $r_2=.08$ ,  $r_3=.11$ , then:

Solving for  $f_1$ ,

$$f_1 = .05$$

Solving for  $f_2$ ,

$$1.05(1+f_2) = 1.08^2 = 1.1664$$

$$f_2 = .1109$$

Solving for  $f_3$ ,

$$(1.05)(1.1109)(1+f_3) = 1.11^3 = 1.3676$$

$$f_3 = .1725$$

Calculating forward and spot rates from bond prices:

In the real world no one gives you the forward rates, you must estimate them. In an efficient market the price of an object equals its present value. So bond prices equal the present value from the bond's cash flows.

How a bond works: A bond has coupons and a face value. The coupons pay off every 6 months for a certain number of years. After all the coupons are paid off the bond is redeemed for its face value.

For example, consider a 10 year bond with a face value of 1000 and 10% coupons. This bond has twenty coupons, each of which pays 50. After 10 years the company redeems the bond for 1000. (Note: 10% is not a true interest rate! Assuming the bond sells for 1000, what is the true interest rate?)

## Bond example:

Mat. Period	Mat. Value	Cou- pon	Price	Name
1	1,000	100	1,000.00	A
2	1,200	60	1,086.49	B
3	800	160	1,004.50	C
4	1,500	50	1,233.21	D
11	300	0	20.00	E
12	10,000	0	512.828	F

Goal: find  $f_1$ , then  $f_2$ , then  $f_3$ , then  $f_4$  and finally  $f_{12}$ , as well as  $r_1$ ,  $r_2$ ,  $r_3$ ,  $r_4$ , and  $r_{11}$  and  $r_{12}$ .

Note: We are finding 6 month forward and spot rates.

Answers:

$$f_1 = .1$$

$$r_1 = .1$$

$$f_2 = .11$$

$$r_2 = .105$$

$$f_3 = .08$$

$$r_3 = .0966$$

$$f_4 = .06$$

$$r_4 = .087$$

$$f_{12} = .3$$

$$r_{11} = .27914 \text{ and } r_{12} = .28086$$

How to find  $f_1$ :

$$PV = 1000 \quad C_1 = 1100$$

$$1000 = \frac{1100}{1 + f_1} \quad \text{so} \quad 1 + f_1 = 1100/1000 = 1.1$$

$$f_1 = .1$$

Since  $r_1 = f_1$ , one has  $r_1 = .1$ .

How to find  $f_2$ :

$$PV = 1086.49 \quad C_1 = 60 \quad C_2 = 1260$$

$$1086.49 = \frac{60}{1 + f_1} + \frac{1260}{(1 + f_1)(1 + f_2)}$$

$$1086.49 = \frac{60}{1.1} + \frac{1260}{1.1(1 + f_2)}$$

Solving for  $f_2$  produces  $f_2 = .11$ .

To find  $f_3$  and  $f_4$  as well as  $r_3$  and  $r_4$  repeat the procedure given above.

To find  $f_{12}$  you must take advantage of the fact that bonds E and F are zero coupon bonds.

Bond E:

$$PV = 20, \quad C_1=C_2=C_3=\dots=C_{10}=0 \quad \text{and} \quad C_{11}=300$$

$$20 = \frac{300}{(1+f_1)(1+f_2)\dots(1+f_{11})}$$

This allows us to write:

$$(1+f_1)(1+f_2)\dots(1+f_{11}) = 15.$$

Since  $(1+r_{11})^{11} = (1+f_1)(1+f_2)\dots(1+f_{11})$  this must mean  $(1+r_{11})^{11} = 15$ , or  $r_{11} = .27914$ .

**Lesson: You can read spot rates right off of a zero coupon bond!**

Bond F:

$$PV = 512.82 \quad C_1=C_2=\dots=C_{11}=0$$

$$C_{12}=10,000$$

$$512.82 = \frac{10,000}{(1+f_1)(1+f_2)\dots(1+f_{11})(1+f_{12})}$$

Now use  $(1+f_1)(1+f_2)\dots(1+f_{11})=15$  to get

$$512.82 = \frac{10000}{15(1+f_{12})}$$

Solving for  $f_{12}$  gives  $f_{12}=.3$ .

For  $r_{12}$  – read it right off of the bond's price and promised payment

$$(1+r_{12})^{12} = (1+f_1)(1+f_2)\dots(1+f_{12}) = 19.5, \text{ or}$$
$$r_{12} = .28086.$$

## Calculating Spot Rates When Bonds Payoff on Fractional Dates

Problem: It is now April 1, and you want to calculate monthly spot rates. You have the following bonds available, each of which has one payment remaining

Date	Payment	Price
April 15	104	99.8
May 15	103	98.7
June 15	106	100.3

What are the monthly spot rates you can calculate?

Answer: You can calculate monthly rates for  $r_{0.5}$ ,  $r_{1.5}$ , and  $r_{2.5}$ . Fractional dates are allowed.

$$99.8 = 104/(1+r_{0.5})^{0.5} \text{ or } r_{0.5} = .0208$$

$$98.7 = 103/(1+r_{1.5})^{1.5} \text{ or } r_{1.5} = .0288$$

$$100.3 = 106/(1+r_{2.5})^{2.5} \text{ or } r_{2.5} = .2236.$$

## A Bond's Yield to Maturity.

Newspapers and other public sources will frequently refer to a bond's yield to maturity (YTM). The YTM solves the following equation for  $y$

$$\text{Bond Price} = \frac{C_1}{1+y} + \frac{C_2}{(1+y)^2} + \frac{C_3}{(1+y)^3} + \dots$$

where the  $C$ 's represent the cash payments from the bond. Some of you may recognize that the YTM equals the **internal rate of return (IRR)** for a bond. Later, the course will discuss the IRR concept in more detail.

The YTM represents the return to an investor that purchases the bond today and holds it until maturity. However, the YTM is *not* an interest rate. It does not satisfy the definition for either a spot or a forward rate. Instead the YTM represents a complex mixture of the spot rates, a mixture that depends upon the size of the coupon payments and the number of periods until maturity.

Topic: Real interest rates, nominal interest rates and inflation.

A nominal cash flow is simply the number of dollars you pay out or receive. A real cash flow is adjusted for inflation. A real dollar always has the same purchasing power. One real dollar today equals one nominal dollar today.

For example: You have \$100, which you invest for 1 year at 10%. Bread sells for \$2.00 today. Inflation over the next year equals 4%.

Next year you receive from the bank 110 *nominal* dollars. Your nominal return equals 10%.

Next year one loaf of bread costs \$2.08, so you can purchase  $110/2.08 = 52.88$  loaves of bread. In real terms you begin with 50 loaves of bread, and earn an additional 2.88 loaves of bread. So in real terms your return equals  $2.88/50 = 5.77\%$ .

In real dollars you begin with 100, earn a return of 5.77%, and thus finish with 105.77 *real* dollars.

Note that  $(1 + \text{real rate})(1 + \text{inflation rate}) = (1 + \text{nominal rate})$ ,  $(1.0577)(1.04) = 1.10$ .

## Rules:

1. Always discount real cash flows with the real interest rate.
2. Always discount nominal cash flows with the nominal interest rate.

The relationship between real interest rates, nominal interest rates and the inflation rate:

Let  $r$  = nominal interest rate,  $r_{\text{real}}$  = real interest rate,  $i$  = inflation rate.

This is incorrect:  $r_{\text{real}} + i = r$

What is correct:

To get PV of  $C_1$  using the real interest rate, first requires you to convert period 1 dollars into period 0 dollars (real dollars) and then discount using the real rate.

Step 1: Convert nominal dollars to real dollars

$$C_{1 \text{ real}} = C_1 / (1+i)$$

Step 2: Discount at the real rate of interest

$$PV = C_{1 \text{ real}} / (1+r_{\text{real}})$$

Combining steps 1 and 2 gives the following present value formula:

$$PV = C_1 / (1+i)(1+r_{\text{real}})$$

So what is the relationship between the real rate, the nominal rate, and the inflation rate?

$$(1+i)(1+r_{\text{real}}) = 1+r$$

Discounting with  
the real rate  
and the inflation  
rate.

Discounting with  
the nominal rate.

Some algebra produces:  $r_{\text{real}} + i + r_{\text{real}}i = r$ .

It is often useful to rearrange this formula so that  $r_{\text{real}} = (r-i)/(1+i)$ .

In most cases  $i$  will be close to zero, so  $r - i$  will roughly approximate the real interest rate.

Examples: If the nominal interest rate equals .12, and the inflation rate is .05 what is the real interest rate?

$$r_{\text{real}} = (.12 - .05)/1.05 = .0667$$

Mistakenly using  $r_{\text{real}} = r - i$  produces a "real" interest rate of .07, a difference of one-third of a percent!

Example: For periods 1 to 3, the nominal interest rate equals .12, and the inflation rate equals .08. You have a project that costs 100 in period zero. In period 1, you expect the project to yield a cash flow of 50. After that, the cash flow grows with inflation in periods 2 and 3. How much is the project worth?

Solution 1: Convert to real rates, and discount real dollars.

$$r_{\text{real}} = (.12 - .08)/(1.08) = .037037.$$

What are the real cash flows? The -100 period zero outflow is a real cash flow. How about the 50 in period 1? No! The 50 in period 1 includes an inflation premium of 8 percent. The real cash flow in period 1 is  $50/1.08 = 46.296$ . Since the profits keep up with inflation, you have a real cash flow of 46.296 in periods 1 through 3.

$$\text{PV} = -100 + 46.296/1.037037 + 46.296/(1.037037)^2 + 46.296/(1.037037)^3 = 29.20.$$

Solution 2: Use the nominal rate, and discount nominal dollars. As given in the problem:

$$r = .12.$$

The nominal cash flows are:  $C_0 = -100$ ,  
 $C_1 = 50$ ,  $C_2 = 50(1.08) = 54$ ,  
 $C_3 = 54(1.08) = 58.32$ .

Therefore,

$$PV = -100 + 50/1.12 + 54/(1.12)^2 + 58.32/(1.12)^3 = 29.20.$$

Notice that both solution methods produce the same answers.

Lesson: It does not matter if you use real rates and real cash flows, or nominal rates and nominal cash flows. The only requirement is that you are consistent.

Perpetuity: A perpetuity is a financial instrument that pays  $C$  dollars per period, forever. If the interest rate is constant and the first payment from the perpetuity arrives in period 1, then the PV of the perpetuity is:

$$PV = \sum_{t=1}^{\infty} \frac{C}{(1+r)^t} = \frac{C}{r}$$

A growing perpetuity pays  $C(1+g)^{t-1}$  per year. For example, if  $C=100$  and  $g=.1$  then you will receive the following payments:

$$C_1 = 100$$

$$C_2 = 100(1.1) = 110$$

$$C_3 = 100(1.1)^2 = 121$$

$$C_4 = 100(1.1)^3 = 133.1$$

The PV of a growing perpetuity is:

$$PV = \sum_{t=1}^{\infty} \frac{C(1+g)^{t-1}}{(1+r)^t} = \frac{C}{r-g}$$

Notice that the growth term acts like a reduction in the interest rate.

An annuity is a financial instrument that pays  $C$  dollars for  $T$  years. It has the following PV formula:

$$PV = \sum_{t=1}^T \frac{C}{(1+r)^t} = \frac{C}{r} \left[ 1 - \frac{1}{(1+r)^T} \right]$$

Easier method: Use the HP-12C

Example: Consider an annuity that pays 50 per year for 20 years. If the interest rate is 8%, what is the PV?

Answer: 490.9

How to enter the problem into the HP-12C

{20}[n]            Enters the number of periods

{8}[i]            Enters the interest rate

{50}[PMT]        Enters the payments

{0}[FV]           Enters a zero future value for the annuity (this is like a bond having a zero face value)

[PV]              Calculates the answer

Note: The calculator displays -490.9 because it considers PMT to be an outflow. Enter {-50} if you want the HP to display a positive number.

## **Example: Mortgage Loan**

You buy a house with a \$120,000 fixed rate mortgage.

30 years to maturity,  
360 equal monthly payments,  
and a quoted interest rate of 9%.

How much do you pay each month?

Mortgage = annuity with 360 monthly payments.  
First payment one month from today.

Monthly = need the monthly interest rate.

The monthly rate on a mortgage is always the quoted rate divided by 12. Monthly rate equals

$$r_m = 0.09 / 12 = 0.0075 \text{ (or 0.75\% per month).}$$

**Key fact:** The remaining balance on a loan at any time equals the present value of the remaining payments, calculated using the loan's interest rate to do all discounting

The remaining balance today must equal the full \$120,000 borrowed, i.e.

$$120,000 = C/1.0075 + C/(1.0075)^2 + \dots + C/(1.0075)^{360}$$

Solving for the monthly payment, C, we find that  
 $C = \$965.55$

## Example: Mortgage Loan continued

Of the first month's payment, how much is interest and how much is principal?

What is the balance remaining on the loan after 3 months? After 10 years?

Monthly rate = 0.75%.

Interest due at the end of the first month is  
 $0.0075 \times 120,000 = \$900.00$

\$900 of the first payment of \$965.55 goes to paying interest.

The remaining  $965.55 - 900 = \$65.55$  goes to paying off the principal.

Loan balance at the end of one month, after making the first payment, is  
 $\$120,000 - 65.55 = \$119,934.55$



Month 2:

Interest charged  $\$119,934.55 \times 0.0075 = \$899.51$ .

$\$899.51$  of month 2's payments goes to paying interest.

$\$965.55 - 899.51 = \$66.04$  pays off principal.

Principal balance after the month 2 payment is  
 $\$119,934.55 - 66.04 = \$119,868.41$

The balance remaining immediately before the month 2 payment is made is

$$\begin{aligned} \$119,934.55 + \$899.51 &= \$119,934.55 \times (1.0075) \\ &= \$120,833.96 \end{aligned}$$

Now month 3:

$$\text{Interest} = \$119,868.41 \times 0.0075 = \$899.01$$

$$\text{So principal} = \$965.55 - 899.01 = \$66.54$$

$$\text{Remaining balance} = \$119,868.41 - 66.54 = \\ \$119,801.88$$

Year 10:

We could continue month by month for 120 periods!

Simpler method -- remember the remaining balance always equals the present value of the remaining payments. Use the loan's interest rate to do all discounting.

After 10 years (120 months) there are  $360 - 120 = 240$  payments remaining. So

$$\text{BAL}_{120} = 965.55/1.0075 + 965.55/(1.0075)^2 + \dots + 965.55/(1.0075)^{240} = \$107,315.69.$$

## Growing Annuity:

A growing annuity pays  $C(1+g)^{t-1}$  per year starting in period 1 for  $T$  periods. Its present value is given by

$$PV = \sum_{t=1}^T C \frac{(1+g)^{t-1}}{(1+r)^t} = \frac{C}{(r-g)} \left[ 1 - \frac{(1+g)^T}{(1+r)^T} \right]$$

$$PV = \sum_{t=1}^T C \frac{(1+g)^{t-1}}{(1+r)^t} = \frac{C}{(r-g)} \left[ 1 - \frac{(1+g)^T}{(1+r)^T} \right]$$

To avoid memorizing the growing annuity formula use the following “trick.” The superscripts associated with the  $(1+g)$  term and the  $(1+r)$  term differ by one period. If they matched you could treat  $(1+g)/(1+r)$  as an “interest rate” and then use the standard annuity formula.

Pull out a  $(1+g)^{-1}$  term to produce

$$PV = \frac{1}{1+g} \sum_{t=1}^T C \frac{(1+g)^t}{(1+r)^t} = \frac{1}{1+g} \sum_{t=1}^T C \left[ \frac{1+g}{1+r} \right]^t.$$

Let  $1/(1+r^*) = (1+g)/(1+r)$  and substitute this into the equation given above,

$$PV = \frac{1}{1+g} \sum_{t=1}^T \frac{C}{(1+r^*)^t}.$$

You can now use your calculator to find the PV. Simply use  $r^*$  for your interest rate, where

$$r^* = (1+r)/(1+g)-1.$$

Growing annuity example.

You have an annuity that pays 100 in period 1. The annuity grows at a rate of 6% per year, and pays off until and including period 10. If the discount rate equals 8% what is the present value?

$$r^* = (1.08)/(1.06) - 1 = .0189$$

$$PV = \frac{1}{1.06} \sum_{t=1}^{10} \frac{100}{(1.0189)^t} = 852.46.$$

If a T period annuity has its first payment (C) arrive in period  $\tau$ , then its value equals

$$\begin{aligned} \text{PV} &= \sum_{t=\tau}^{T+\tau-1} C \frac{(1+g)^{t-\tau}}{(1+r)^t} \\ &= \frac{C}{(r-g)(1+r)^{(\tau-1)}} \left[ 1 - \frac{(1+g)^T}{(1+r)^T} \right] \end{aligned}$$

Example: An insurance company offers a retirement annuity that pays 100 per year for 15 years and sells for 806.07. What is the interest rate?

$$806.07 = \sum_{t=1}^{15} \frac{100}{(1+r)^t}$$

Solving for  $r$  gives the yield to maturity. Answer:  
 $r = .09$

## How to find $r$ with the HP-12C

{806.07}[CHS][PV]	Cost of the annuity
{100}[PMT]	Payments
{15}[n]	Number of payments
[i]	Finds interest rate

Example: The retirement problem.

You are 25 years old. You will retire at the age of 65, and die when you are 90. Your goal is to have \$50,000, in today's dollars, a year to spend during your final years. (ie. You need \$50,000 in your 66th year, through and including your 90th year.) Beginning next year, you will save a fixed amount, in today's dollars, every year until retirement. If the return on your investments equals 12%, and the inflation rate equals 8%, how much do you need to invest each year?

Since the problem is couched in terms of real dollars, it is easier to use real interest rates.

As always  $r_{\text{real}}$  = real interest rate,  $r$  = nominal interest rate, and  $i$  = inflation rate.

From our formula:  $r_{\text{real}} + i + r_{\text{real}}i = r$ .

$$r_{\text{real}} = (.12 - .08) / 1.08 = .037.$$

To answer the question, consider the problem from your perspective at age 65. You want an annuity that pays \$50,000 per year starting in one year, and that lasts 25 periods. What is its present value when you are 65.

$$\begin{aligned} & \text{PV}(\text{when you are 65}) \\ &= 50,000/1.037 + \dots + 50,000/(1.037)^{25} \\ &= 806,473. \end{aligned}$$

This means you must have saved \$806,473 in real dollars by the time you reach 65.

Now what is the present value of the annuity today? You are 65 in period 40, so the PV of the annuity equals  $806,473/(1.037)^{40} = 188,556$ .

Goal: Find constant payments (C) that satisfy the following equation.

$$188,556 = C/1.037 + C/(1.037)^2 + \dots + C/(1.037)^{40}$$

Using a calculator to solve for C yields:

$$C = 9,105.$$

So you need to put away \$9,105 per year, every year until you retire. Good luck!

Solving for C on the HP-12C.

{188556}[CHS][PV] Present value.

{40}[n] # of payments.

{3.7}[i] Interest rate.

{0}[FV] Zero future value.

[PMT] Find C.

Topic: Stock Pricing

Define:

$DIV_t$  as the dividend in year t.

$P_t$  as the price in year t.

Then the price of a stock today (period zero) is:

$$P_0 = \frac{DIV_1 + P_1}{1 + r}$$

One can rearrange the above equation to solve for  $r$ :

$$r = \frac{\text{DIV}_1 + P_1 - P_0}{P_0}$$

In this case  $r$  is called the market capitalization rate. It is the expected return on a firm's equity.

Why there is no such thing as an investor with a short term horizon. (Or why all investors are long term investors.)

Given  $r$ , what is the stock's value in period 0?

$$P_0 = \frac{DIV_1 + P_1}{1 + r}$$

In order to calculate  $P_0$  we now need to find  $P_1$ . How is  $P_1$  determined? In a manner similar to  $P_0$ :

$$P_1 = \frac{\text{DIV}_2 + P_2}{1 + r}$$

Now plug the formula for  $P_1$  into the formula for  $P_0$  to get

$$P_0 = \frac{\text{DIV}_1}{1 + r} + \frac{\text{DIV}_2 + P_2}{(1 + r)^2}$$

New problem. How is  $P_2$  determined? Well,  $P_2$  is determined in a manner similar to  $P_0$  and  $P_1$ .

$$P_2 = \frac{\text{DIV}_3 + P_3}{1 + r}$$

Given the formula for  $P_2$ , we can now plug it into our formula for  $P_0$  to get

$$P_0 = \frac{\text{DIV}_1}{1 + r} + \frac{\text{DIV}_2}{(1 + r)^2} + \frac{\text{DIV}_3 + P_3}{(1 + r)^3}$$

Continuing through time, one eventually obtains:

$$P_0 = \sum_{t=1}^{\infty} \frac{DIV_t}{(1+r)^t}$$

What this implies is that every investor is concerned not only with short term profits, but the present value of the entire series of dividends. This holds no matter how long (or short) a person expects to hold a security!

Using the perpetuity and annuity formulas to price stocks.

Example: The Gizmo Company has strong short term growth prospects. In period 1, it will pay a dividend of \$5 per share, and the dividend will grow at a rate of 10% per period until the period 5 dividend. After that no further growth is expected. If the market discounts the dividend stream at 15%, how much is a share worth?

The stock's value consists of two parts. Part one is a growing annuity in periods 1 through 5. Part two is a perpetuity, whose first payment occurs in period 6.

## Growing Annuity:

$$\text{PV(annuity)} = \frac{5}{1.15} + \frac{5(1.1)}{(1.15)^2} + \dots + \frac{5(1.1)^4}{(1.15)^5}.$$

In principle you can enter all this into your calculator and get the answer. This is very time consuming. Alternatively we know an explicit formula. However, if you do not want to remember the formula, the following is a useful trick.

First, pull out a  $1/1.1$  to get

$$\text{PV} = \frac{1}{1.1} \left[ 5 \frac{1.1}{1.15} + 5 \left( \frac{1.1}{1.15} \right)^2 + \dots + 5 \left( \frac{1.1}{1.15} \right)^5 \right]$$

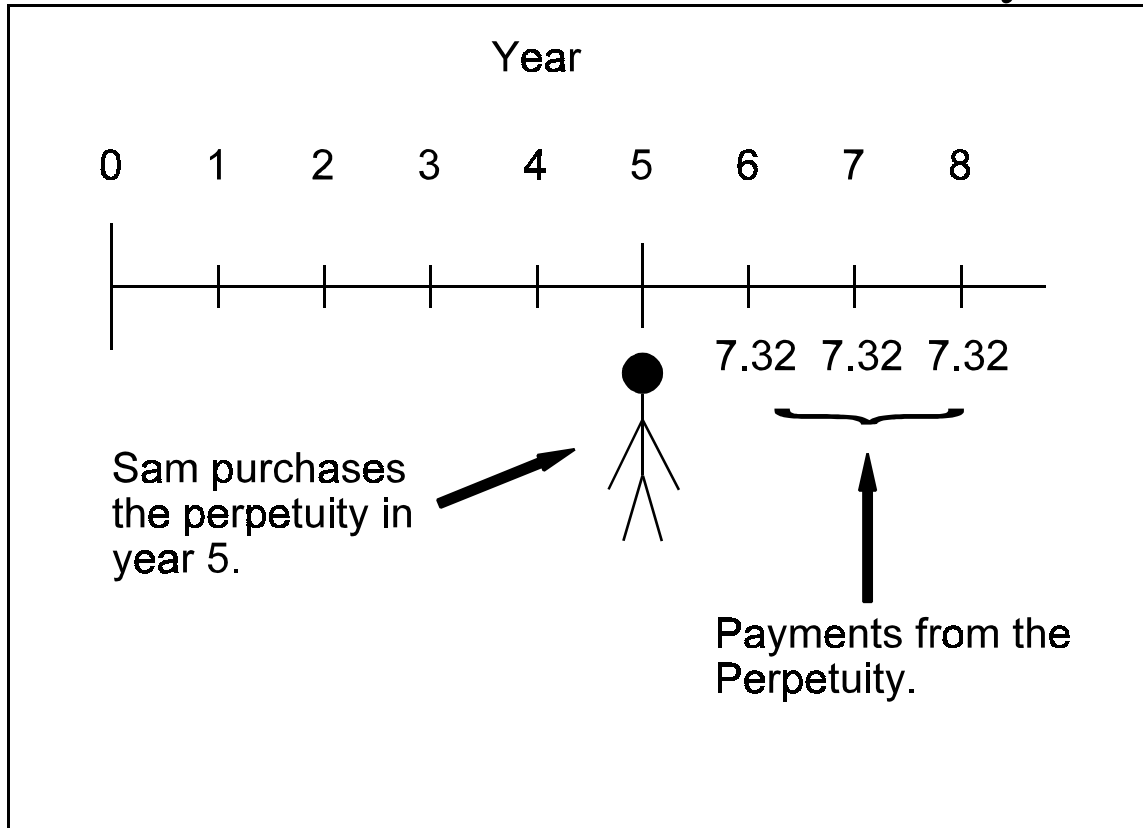
Now,  $1.1/1.15 = 1/\{1.15/1.1\} = 1/1.045$ . So the part in brackets reduces to a 5 year annuity, with a \$5 per period payment, discounted at 4.5%. Entering this information into your calculator yields:

$$\text{PV}(\text{annuity}) = 21.92/1.1 = 19.93.$$

Check that the formula gives the same answer.

Perpetuity: The perpetuity part pays  $5(1.1)^4=7.32$ , beginning in year 6.

To calculate the value of the perpetuity first consider how much it will be worth in year 5.



If Sam buys the perpetuity in period 5, from his perspective the first payment occurs *one* period later. So, in period 5, Sam will pay  $7.32/.15 = 48.80$  for the perpetuity.

If Sam buys the stock for 48.80, in period 5, what is the perpetuity worth today? Answer:  $48.80/(1.15)^5 = 24.26$ .

The stock's value today equals:

$$\begin{array}{rclcl} \text{PV(annuity)} & + & \text{PV(perpetuity)} & = & \\ 19.93 & + & 24.26 & = & 44.19. \end{array}$$

Simple dividend growth model of stock prices:

Assume a stock's dividend will grow at a rate  $g$ . Then using the perpetuity formula the stock's price today can be expressed as:

$$P_0 = \text{DIV}_1 / (r - g)$$

In real life you know  $P_0$ ,  $\text{DIV}_1$  and can sometimes estimate  $g$ . When this occurs you can rearrange the above equation to produce an expression for the market capitalization rate.

$$r = \text{DIV}_1 / P_0 + g$$

This is a simple way to estimate  $r$ .

Example: Stock ABC will pay a dividend of \$2 per share next period. The dividend grows at 3% a year and presently the stock sells for \$20. What is the market capitalization rate?

$$r = 2/20 + .03 = .13$$

The next issue is how to estimate  $g$ . A crude method is to use the following model.

Assume that every year a firm pays out a constant percentage of its earnings and invests the rest. In addition, assume the return on equity is constant through time.

Step 1: Estimate the "plowback" and "payout" ratios. The plowback ratio is the percentage of the company's profits that are reinvested every year. The payout ratio is the percentage of the firm's profits that are returned to the investors every year.

Define  $EPS_t$  as the firm's earnings per share in year  $t$ . Then

$$\begin{aligned}\text{Plowback ratio} &= 1 - \text{Payout ratio} \\ &= 1 - \text{DIV}_1/\text{EPS}_1\end{aligned}$$

Step 2: Estimate the return on equity (ROE)

$$\text{ROE} = \text{EPS}_1/\text{Book Value Per Share}$$

Step 3: Calculate  $g$ , where the dividend growth rate is estimated by

$$g = \text{Plowback ratio} \times \text{ROE}$$

Warning: The above formulas require that the Plowback ratio, ROE, and the discount rate remain constant forever. If they do not, the formula will not necessarily provide accurate estimates. Read questions carefully to ensure the conditions for the formulas are met before trying to find an answer.

Example: ABC Co. has earnings per share of \$10. It pays a \$2 dividend and has a book value per share of \$100. What is  $g$ ?

$$\text{Plowback ratio} = 1 - 2/10 = .8$$

$$\text{ROE} = 10/100 = .1$$

$$g = .8 \times .1 = .08$$

Example: At present DEF Co. will pay an \$8 dividend per share on earnings per share of \$10. Its present book value per share equals \$100. Using the perpetual growth model described above, the market values the stock at \$110. The firm now discovers a new investment opportunity, with the same return as its previous projects. It therefore alters its plowback ratio, and as a result next period's dividend falls to \$6. What will the stock sell for?

To answer questions like this begin with a list of what you know.

Dividend	\$8
Earnings	\$10
Book value	\$100
Stock price	\$110

What is missing? The plowback ratio, return on equity, the growth rate, and the discount rate.

$$\text{Plowback ratio} = 1 - \text{dividend/earnings} = .2$$

$$\text{ROE} = \text{Earnings/Book value} = .10$$

$$g = \text{Plowback ratio} \times \text{ROE} = .2 \times .10 = .02$$

$$r = \text{DIV}_1/P_0 + g = 8/110 + .02 = .0927$$

You now know everything there is to know about the firm.

To answer the question we need to calculate how the factors in the model change.

$$\text{Plowback ratio} = 1 - 6/10 = .4$$

What is the ROE after the new investment policy?  
The question states that the return on the new project equals the return on the previous projects.  
Is this important? Why?

Because the ROE on the new projects equals that on the previous projects we can continue to use  $ROE = \text{Earnings}/\text{Book} = .1$  in our calculations. If the new projects have a different ROE, then as time goes on the firm's ROE changes, and we cannot use our formulas.

$$g = .4 \times .1 = .04$$

$$P_0 = \text{DIV}_1 / (r - g) = 6 / (.0927 - .04) = 113.85$$

Notice the stock price has gone up 3.85. Why? Because the cost of capital is only 9.27%, but the new investment returns 10%. This leads to a gain for the shareholders.

Which of the following firms is worth the most if the market capitalization rate is 10%?

Firm A: Book = 100,  $EPS_1 = 10$ ,  $DIV_1 = 10$   
Plowback ratio = 0,  $ROE = 10/100$ ,  $g = 0$

Firm B: Book = 100,  $EPS_1 = 10$ ,  $DIV_1 = 6$   
Plowback ratio =  $4/10$ ,  $ROE = 10/100$ ,  
 $g = 4/100$

Firm C: Book = 150,  $EPS_1 = 10$ ,  $DIV_1 = 2$   
Plowback ratio =  $8/10$ ,  $ROE = 10/150$ ,  
 $g = 5.33/100$

Notice: For firm A the first dividend is the highest but the growth rate is the lowest, while firm C has the lowest initial dividend but the highest growth rate.

Using our formulas  $P_0 = \text{DIV}_1 / (r - g)$  and  $g = \text{Plowback ratio} \times \text{ROE}$  we get

$$P_0(\text{A}) = 10 / .1 = 100$$

$$P_0(\text{B}) = 6 / (.1 - .04) = 100$$

$$P_0(\text{C}) = 2 / (.1 - .053) = 42.6$$

So firm A and B have the same value, while firm C is worth the least. Why?

For firms A and B the ROE is the same as the market capitalization rate. Thus, all their investments have a NPV of zero. That is a one dollar investment gives back one dollar in value. Lesson: Investing in zero net present value projects cannot change a firm's value!

Why does firm C have the lowest value?

Because its investments have *NEGATIVE* NPVs. The firm invests in projects which return only .066 per year yet the cash flows are discounted at .1 per year. Every dollar they invest pays only 66 cents in present value! (To see this figure out what .066 per year forever is worth and compare this to the 1 dollar you must invest to receive it.)

## PVGO: Present Value of Growth Opportunities

Consider the following firm:

$$\text{EPS}_1 = 10 \text{ and } r = .1$$

Suppose  $\text{DIV}_1=10$ , then the Plowback ratio = 0 and thus one expects  $\text{EPS}_2=10$ . Without any new investment earnings should not change. Now suppose  $\text{DIV}_2=10$ , then  $\text{EPS}_3=10$  etc. Thus, if the firm pays a dividend of 10 per year forever:

$$P_0 = 10/.1 = 100$$

Now suppose the firm invests \$2 in year 1 on a project that returns 10% per year forever. After year 1, it pays everything out as a dividend. Then the following cash flows result:

$$\text{EPS}_1 = 10 \qquad \text{DIV}_1 = 8$$

$$\text{EPS}_2 = 10.2 \qquad \text{DIV}_2 = 10.2$$

$$\text{EPS}_3 = 10.2 \qquad \text{DIV}_3 = 10.2$$

$$P_0 = \frac{10 - 2}{1.1} + \frac{10 + .2}{(1.1)^2} + \frac{10 + .2}{(1.1)^3} + \dots$$

$$P_0 = \left( \frac{10}{1.1} + \frac{10}{(1.1)^2} + \frac{10}{(1.1)^3} + \dots \right) + \left( \frac{-2}{1.1} + \frac{.2}{(1.1)^2} + \frac{.2}{(1.1)^3} + \dots \right)$$

PV of the firm's  
NO growth  
component.

+

Growth  
component's PV  
(PVGO).

$$P_0 = \frac{10}{.1} - \frac{2}{1.1} + \frac{.2}{.1(1.1)} = 100$$

The investment did not alter the firm's value. Why? Because the NPV of the investment is zero. So PVGO equals zero in this case.

Another example: Now suppose the firm invests \$2 in year 1 in a project that returns 20% per year forever. After year 1 all earnings are paid out as dividends. Thus,

$$\text{EPS}_1 = 10$$

$$\text{DIV}_1 = 8$$

$$\text{EPS}_2 = 10.4$$

$$\text{DIV}_2 = 10.4$$

$$\text{EPS}_3 = 10.4$$

$$\text{DIV}_3 = 10.4$$

$$P_0 = \frac{10 - 2}{1.1} + \frac{10 + .4}{(1.1)^2} + \frac{10 + .4}{(1.1)^3} + \dots$$

$$P_0 = \left( \frac{10}{1.1} + \frac{10}{(1.1)^2} + \frac{10}{(1.1)^3} + \dots \right) + \left( \frac{-2}{1.1} + \frac{.4}{(1.1)^2} + \frac{.4}{(1.1)^3} + \dots \right)$$

PV of the firm's  
NO growth  
component.

+

Growth  
component's PV  
(PVGO)

$$P_0 = \frac{10}{.1} - \frac{2}{1.1} + \frac{.4}{.1(1.1)} = 100 + \frac{2}{1.1}$$

In this case  $PVGO = 2/1.1 > 0$ . Thus, the investment has increased the firm's value.

One can express a firm's stock price as:

$$P_0 = \text{PV of the no growth component of the firm} \\ + \text{PVGO}$$

Note: This formula is very general. It holds when the interest rate, growth rates and other factors vary over time.

What is the PV of the no growth component?

Assume that if the firm does not invest it will not grow. Thus, setting  $DIV_t = EPS_t$  results in a constant dividend forever, implying

PV of the no growth component of the firm =  $EPS_1/r$

We can now obtain  $P_0$  in terms of PVGO,  $EPS_1$  and  $r$ :

$$P_0 = EPS_1/r + PVGO$$

Rearranging and solving for the earnings-price ratio produces:

$$\frac{\text{EPS}_1}{P_0} = r \left[ 1 - \frac{\text{PVGO}}{P_0} \right]$$

If  $\text{PVGO}=0$  the earnings-price ratio equals  $r$ . The earnings-price ratio is often used as a measure of a firm's market capitalization rate.

Warning: Using the earnings-price ratio to estimate  $r$  is dangerous! If  $\text{PVGO}>0$  then  $\text{EPS}_1/P_0 < r$ . Generally, the earnings-price ratio will *underestimate*  $r$ .