

Closed-End Fund Discounts in a Rational Agent Economy

by

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Abstract

Nearly any standard financial model concludes that two assets with identical cash flows must sell for the same price. Closed-end mutual fund company share prices seem to violate this fundamental tenant. Even when one considers several standard frictions, such as taxes and agency costs, classical financial models cannot easily explain the large persistent discounts found in the data. This paper shows that in an otherwise frictionless market, if production capacity varies randomly over time and agents possess finite lives a closed-end mutual fund's stock price may not track its net asset value. The analysis provides a number of conditions under which these discrepancies will lead to the existence of systematic discounts for the mutual fund's shares. In addition, the model provides predictions regarding the correlation between current closed-end fund discounts and current changes in stock prices and future changes in corporate productivity. The same parameter values that lead to systematic discounts also lead to other fund price characteristics that resemble many of the results found within empirical studies. Finally, the results in this paper may also be of interest to practitioners since they apply to zero dividend portfolios in general and not just those created by a positions that involve closed-end funds. In a typical financial model such portfolios must always sell for a zero price. Thus, when they do not traditional models provide very little guidance regarding the risks one might take on by investing in them. In contrast, since the model in this paper shows that zero dividend portfolios can sell for a non-zero amount, it also provides a theoretical bases for estimating the risk-return tradeoff such portfolios offer to investors.

Closed-end mutual funds tend to sell at a discount relative to their net asset value. How can this be? Any reasonable theory of capital markets should rule out arbitrage, and closed-end fund discounts appear to provide arbitrage opportunities. Malkiel (1977) put forth a number of frictions (such as taxes and agency costs) that might explain observed closed-end fund discounts. However, he shows that while these frictions can account for some of the observed discount they cannot account for all of it.¹ Lee, Shleifer and Thaler (1990) make the additional point that even if taxes and agency costs did explain the observed magnitude of closed-end fund discounts, there still remain a number of outstanding issues regarding time trends in the data that neither agency costs nor taxes seem well equipped to deal with.

In response to the inability of the standard financial model to fully explain the closed-end fund data, two alternative explanations have been proposed. In Lee, Shleifer, and Thaler (1991) small irrational agents create the discounts by trading on correlated sentiment. In Chordia and Swaminathan (1997) frictions including asymmetric information and market segmentation lead to closed-end fund discounts. While both of these models undoubtedly help solve this puzzle, they do represent rather significant departures from the standard financial markets model in which rational agents trade in a frictionless environment. This paper argues that it may be too soon to give up on the standard capital markets model since a rather close cousin of that model can explain both observed closed-end fund discounts and its associated properties.

By making two relatively minor modifications to the standard capital asset market's model one can explain many of the empirical regularities and produce some, as yet, untested predictions. First, this paper drops the assumption that people live forever and replaces it with agents that have

¹Additional work on the relationship between taxes and observed closed-end fund discounts has since been conducted by Brickley, Manaster, and Schallheim (1991), and Pontiff (1995). Both papers provide mixed evidence regarding the empirical validity of this hypothesis. With regard to agency costs, Barclay, Holderness, and Pontiff (1993) provide additional support regarding their influence on the discount. More recently, Pontiff (1996) trading costs may explain part of the discount's magnitude and behavior.

finite lives. This rather simple, and realistic, change does a great deal to alter an investor's equilibrium behavior. When agents only live for a finite amount of time they only care about price movements and dividend cash flows that will occur before their deaths. Of course this point is not new and can be found in articles such as Dow and Gorton (1994) and Holden and Subrahmanyam (1996). Second, the model assumes that productive corporate capital varies randomly through time. This captures the idea that while agents may know something about the quantity of capital equipment in use by various firms they do not know precisely what these levels will be in the future. By adding both features to a frictionless multi-asset model a second equilibrium emerges.²

Intuitively, the model shows that closed-end fund discounts can occur if people simply believe that they can occur. Arbitrage arguments, as to why discounts cannot exist, rest on the idea that if a discount does appear investors will try to take advantage of the situation by going infinitely long the fund and short the underlying securities. But imagine that you discover a closed-end fund that currently sells for less than its net asset value. Would you willingly take an infinitely long position in the fund and a short position in the underlying stocks? Not if you had to close out the position by some finite date, and you thought that the discount might widen prior to then. Instead you might take a small position depending upon your beliefs regarding the risk-return tradeoff. But this means that traders submit finite (rather than infinite) demands in response to discounts. If asset supplies vary then that variation will induce the discount to fluctuate over time, fulfilling people's beliefs.

The model's analysis not only shows that discounts can exist within a fairly classical model, but also that the existence of discounts can be linked to a number of other related phenomena. The same parameter values of the model that produce closed-end fund discounts also produce a

²The fact that overlapping generations models can produce multiple equilibria goes back to Azariadis (1981).

number of other patterns, many of which have been documented in empirical studies. These include: (1) closed-end fund prices are more volatile than the underlying net asset value (Pontiff (1997)), (2) closed-end fund discounts are mean reverting (Thompson (1978), and Pontiff (1995)), (3) closed-end fund discounts widen when concurrent corporate productivity (perhaps proxied for by stock prices) goes up unexpectedly (Malkiel (1977) and Pontiff (1995)), and (4) closed-end fund discounts are negatively associated with future aggregate economic activity (Swaminathan (1996)). In addition the model also generates a number of new, and untested, predictions regarding cross-sectional patterns between the time trend in the discount and changes within the real economy. The findings with regard to corporate productivity are summarized in the following table.

Predictions on the Relationship Between Closed-End Fund Discounts and Corporate Productivity	Empirical Work ● = Verified, ✕ = Not Yet Tested
If the discount is correlated with small firm returns then it is also correlated with the stock of small firm capital.	✕
Larger discounts associated with funds that hold securities in industries with relatively slow mean reverting capital supplies.	✕
Discount positively correlated with current corporate production.	●
Discount should be stronger for funds holding industries with capital bases that mean revert relatively slowly.	✕
Negative correlation between closed-end fund discounts and future corporate production.	●
Stronger negative correlation between closed-end funds and future corporate production for funds that hold industries with capital bases that mean revert relatively slowly.	✕

At its heart the closed-end fund puzzle really asks how can an “arbitrage” portfolio sell for a non-zero amount. Thus, any model about the closed-end fund puzzle can also be applied more generally to other arbitrage portfolios. In light of the recent financial breakdowns in the hedge

fund industry this seems to be a particularly timely issue. Hedge funds generally take offsetting long and short positions, and thus they collapse when changes in relative market prices move against them. The model presented in this paper offers a possible explanation as to where they went wrong. In a typical financial model arbitrage portfolios must always sell for a price of zero. Thus, those models provide very little guidance regarding the price path of these portfolios should prices diverge from their assigned equilibrium value. In contrast, since the model presented here finds that arbitrage portfolios need not sell for exactly zero it also offers guidance about the possible price path.³ This in turn can be used by practitioners to help gauge the potential risks and returns that the market may offer in such cases.

On a methodological note this paper provides a justification for the mathematical structure found in “random share supply” models based upon first principles. The model presented here begins with the assumption that corporate productive capacity changes randomly over time, and that these changes arise from the efforts of individuals whose remuneration is determined by a competitive market. If one now considers investor demand over units of capital rather than shares of stock, the resulting equilibrium conditions are equivalent to what one typically begins with in a random share supply model. The advantage of this paper’s derivation is that it helps to firmly distinguish random share supply models from “noise trader” models. In a noise trader model, the market phenomena induced by the noise traders will be orthogonal to real changes in the productive capacity of the economy. In contrast, random share supply models (when derived from

³This is not to say that no other models exist in which “arbitrage” portfolios sell for nonzero amounts. Shleifer and Vishny (1997) discuss how capital constraints can produce arbitrage opportunities that the market cannot eliminate. In contrast this paper shows that arbitrage portfolios may sell for nonzero amounts even in the absence of such constraints. Alternatively, one can seek guidance from bubble models such as Jackson (1994). Bubble based explanations differ from the present model in both their assumptions and results. For example, bubbles must have an expected growth rate equal to the interest rate. In the model presented here the price of an arbitrage portfolio will not on average display such growth.

first principles) *tie* the induced market phenomena to changes in corporate production. This is what allows the current paper to both reconcile the set of currently documented correlations between closed-end fund discounts and corporate productivity, and to produce new and untested predictions in this area.

The paper's structure is as follows. Section 1 presents the model. Part 1.1 lays out the mathematical details describing the economy. Part 1.2 calculates the equilibrium conditions for the model's parameters. Next part 1.4 outlines the basic properties of the solution with regard to closed-end fund discounts, and provides some empirical predictions. Part 1.5 examines the relationship between closed-end fund discounts and future economic activity within the corporate sector. Section 2 discusses the paper's relationship to the theoretical literature. Section 3 concludes. Appendix A discusses the stability of the different equilibria, and Appendix B contains most of the proofs.

1 The Model

1.1 Description of the Economy

The economy progresses through time via a sequence of overlapping generations. People live for two periods, and consume only in their last period of life. Their preferences over consumption are represented via a negative exponential utility function with risk aversion parameter θ . At the start of period t a new generation of individuals are born. For simplicity, the model assumes that the population's size does not vary over time, with each generation being of size "one."

There are two parts to the economy a "real" side and a "financial" side. The real side represents the actual production processes in the economy such as plant and equipment which the paper calls the corporate capital stock. These real assets produce cash flows and in combination

make up firms. The financial side of the economy represents the place where claims against these assets are traded.

Firms use their capital stock to produce cash flows. Denote the period t capital stock via the $k \times 1$ vector $N_t = (n_{t1}, n_{t2}, \dots, n_{tk})'$ and the corresponding dividend vector by $D_t = (d_{t1}, d_{t2}, \dots, d_{tk})'$.⁴ For simplicity let each element of N_t represent the capital stock of a particular firm.⁵ Thus, firm i 's cash flow production in period $t+1$ equals $n_{it}d_{t+1,i}$. Not all of these cash flows go to the investors, rather some of them go to pay off labor and their wages are represented by the $k \times 1$ vector $G_t = (g_{t1}, g_{t2}, \dots, g_{tk})'$. Therefore, the period t owners of firm i receive a aggregate dividend payment of $n_{it}d_{t+1,i} - g_{t+1,i}$ the next period.

Upon birth the period t generation is endowed with a $k \times 1$ vector of "human capital." When people enter the workforce they sell their endowment of human capital to the corporate sector which then turns it into corporate capital. One can think of the conversion process this way: people are born with skills and when in the employ of firms they use their skills to construct capital items like buildings and machinery. Once the capital items are built they then become part of the corporate capital base. The model assumes that the period t endowment of human capital will be transformed into an amount of corporate capital equal to the vector $A(\bar{N} - N_{t-1}) + \eta_t$. Within this formula, A equals a $k \times k$ constant matrix with elements a_{ij} , and \bar{N} a $k \times 1$ constant vector all of

⁴In keeping with notational tradition the paper uses capital letters to represent a matrix and the corresponding small letter to represent its individual scalar elements.

⁵The assumption that each element of N_t represents the capital stock for a particular firm is made for simplicity. By making N_t a random matrix one can then treat each element of the capital stock as a factor that goes into a particular firm's production. Each row of N_t then designates the quantity of each factor used by a particular firm. This framework also allows for non-square N_t which would enable the model to examine economies where there exist more capital factors than firms or visa-versa. Since these generalizations do not alter the paper's primary conclusions they are not explored any further within this text.

whose elements are nonnegative.⁶ Throughout, the paper assumes that $I-A$ is a positive definite matrix with eigenvalues between zero and one, where I is a $k \times k$ identity matrix. The η_t term equals a normally distributed $k \times 1$ vector with variance-covariance matrix Σ_η .⁷

The human capital endowment equation captures two elements of the real economy. First, the mean reversion term $A(\bar{N} - N_{t-1})$ proxies for the idea that people will direct their energies via schooling and other activities towards ventures where they can get the best return. Thus, people will tend to add capital to industries where it is in short supply (and thus the rewards higher) and away from industries where capital is in abundant supply (and thus the rewards lower). For example, imagine that there exists a relative shortage of people with skills associated with biology in comparison with those skilled in history. Then in the aggregate one would expect people to direct their education towards biology and away from history. The matrix A captures this phenomena without imposing the considerable mathematical complexity that would be involved by allowing people to endogenously redirect their endowments at some exogenously specified cost. While such a generalization may yet provide insights into some interesting issues, the results that form the primary focus of this study seem unlikely to change since they hold for any matrix A that meets the restrictions given above. Endogenously calculating A from the forces directing the mean reversion of the security supplies would simply place some additional restrictions on A based upon some other exogenously given cost matrix. Thus, in an appeal to simplicity the current analysis

⁶The assumption on \bar{N} is just a normalization. If an element i is negative simply multiply it as well as n_{it} , and d_{it} by minus one. The firm will then produce the same net output $n_{it}d_{t+1,i}$ in the following period.

⁷At the expense of algebraic complexity one can add another drift term that does not interact with past values of N_t without altering any of the paper's basic conclusions. This additional drift term would cause the per capita capital stock to exhibit either long run growth or decay over time.

simply takes A as given.⁸ The second facet of the economy captured by the endowment equation comes in via the random vector η_t . While people may be able to forecast future corporate capital stocks, those forecasts are by no means perfect, and this imperfection enters the model via η_t .

Based upon the above assumptions the corporate capital stock evolves through time via the following equation

$$N_{t+1} = N_t + A(\bar{N} - N_t) + \eta_{t+1}. \quad (1)$$

In addition, to the risky corporate capital represented by N_t there also exists a risk free asset. Following standard practice in the literature, the riskless asset is in infinitely elastic supply and used by the population to either borrow from future consumption or store current consumption at no cost. For notational simplicity, the paper normalizes the price of the risk free asset so that it always equals one.

Randomness enters the economy via both the capital stock and the dividend paid per unit of capital. Over time the productivity of the capital supply evolves via

$$D_{t+1} = D_t + \delta_{t+1} \quad (2)$$

where δ_t represents a $k \times 1$ vector of normally distributed shocks with mean zero and variance-covariance matrix Σ_δ .⁹ For simplicity, the model assumes that the δ_t and η_t shocks are independent from each other and that there does not exist any intertemporal correlation across

⁸Elimination of the mean reversion in the supply process can be accomplished by simply setting A to a matrix of zeros. Doing so will leave most of the model's conclusions unchanged, including many results on the price and volatility of closed-end funds relative to their underlying assets. However, without a mean reverting supply process the model will no longer produce a mean reverting discount, and the comparative statics concerning the impact of A on the equilibrium will obviously be lost.

⁹Again, one can generalize this process to include terms that induce mean reversion and/or growth in the dividend process without altering the paper's basic conclusions.

shocks either. The matrix Σ_δ may be singular if some forms of capital produce cash flows identical to other forms of capital.

1.2 Equilibrium

With the elements of the economy in place one can now consider the problems faced by investors within the model. Based upon the above notation, investor i 's final period consumption equals

$$w_{t+1}(i) = F_t'(i) \left[N_{t+1} \cdot P_{t+1} + N_t \cdot D_{t+1} - G_{t+1} \right] + (1+r)\beta_t(i) \quad (3)$$

where the scalar w_t equals period t wealth, the $k \times 1$ vector $F_t(i)$ equals the investor's period t security demands, and β_t the investor's period t demand for the riskless asset. Since each firm always has one share outstanding F equals the fraction of the firm demanded by an investor. The “ \cdot ” symbol stands for the dot or element by element product of two vectors. Variables without a “ (i) ” represent aggregates across the population. On the left hand side of equation (3) the first term equals the cash inflow the investor receives from his investment in the stock market and the second term the cash inflows from his investment in the riskless security. Inside the square brackets the $N_{t+1} \cdot P_{t+1}$ term produces a vector that contains the market price of each firm. To see this recall that N_t equals the vector of corporate capital and P_t the market price per unit of corporate capital vector. Thus, their dot product produces a vector with the market price of each firm. The $N_t \cdot D_{t+1}$ term is a vector that contains the pre-wage cash flow produced by each firm. Finally, the G_{t+1} term equals the wages paid in exchange for the capital additions that transformed N_t into N_{t+1} . Thus differencing the last two terms in the square brackets yields a vector containing the dividends paid out by the firms within the economy.

To remain close to the standard asset pricing model, there are no frictions in either the

form of transactions costs or taxes.¹⁰ Everyone can trade every available security. Thus an investor's budget constraint can be written as

$$F'_t(i)N_t \cdot P_t + \beta_t(i) = G_t(i) + \beta_{t-1}(i). \quad (4)$$

Use this to eliminate β_t from equation (3), thereby producing

$$w_{t+1}(i) = F'_t(i)N_{t+1} \cdot P_{t+1} - (1+r)N_t \cdot P_t + N_t \cdot D_{t+1} - G_{t+1} + (1+r)w_t(i), \quad (5)$$

where $w_t(i)$ equals the investor's initial cash position from the sale of his bond and human capital endowment. Given the above wealth equation each trader will seek to maximize his expected utility from terminal consumption by selecting an appropriate demand vector $F_t(i)$.

To further simplify (5) recall that the model assumes labor sells its services in a perfectly competitive market.¹¹ This implies that

$$G_{t+1} = \left[A(\bar{N} - N_t) + \eta_{t+1} \right] \cdot P_{t+1} = \left[N_{t+1} - N_t \right] \cdot P_{t+1}. \quad (6)$$

Using the far right term to eliminate G_{t+1} from (5) and yields

$$w_{t+1}(i) = \left[F'_t(i) \cdot N_t \right] \left[P_{t+1} - (1+r)P_t + D_{t+1} \right] + (1+r)w_t(i). \quad (7)$$

after some minor algebra to move the N_t term outside the brackets.

To eliminate the dot product in (7), define $X_t(i) = F_t(i) \cdot N_t$, in which case the wealth equation

¹⁰A related overlapping generations exponential-normal model can be found in Vayanos (1996). His paper examines how the addition of transactions costs in a fixed supply model alter equilibrium prices. Based upon his results it appears that the addition of transactions costs to such models can produce a number of interesting predictions. However, since this paper seeks to find out how much of the closed-end puzzle can be explained within a frictionless economy these issues are not explored further.

¹¹Other assumptions concerning labor's compensation will produce similar results within the model, although at the potential expense of additional algebraic complexity.

further reduces to

$$w_{t+1}(i) = X_t(i)' [P_{t+1} - (1+r)P_t + D_{t+1}] + (1+r)w_t(i) \quad (8)$$

with the $X_t(i)$ as the new control variable. Notice that equation (8) looks like the wealth equation in a random share supply model, with X as the typical control variable. The derivation of this equation shows that the equilibrium conditions of such a model are intimately tied to the state of the real economy and its current level of production. Based upon the arguments above, one need not interpret “share” demands in a random share supply model as literally demands over shares, since a firm’s total production equals nd and not just d . Rather share demands can be interpreted as demands over real assets. Market clearing then occurs when aggregate demand for real assets equals the aggregate supply of real assets.

If in equilibrium the vector P_{t+1} is normally distributed (given the information available at time t) the exponential utility function and the normality of the other unknowns in the system imply that the investor faces a standard mean-variance optimization problem. Given the wealth equation (8) the investor will seek to

$$\max_{X_t(i)} X_t(i)' [E_t(P_{t+1}) - P_t + E_t(D_{t+1}) - rP_t] + (1+r)w_t(i) - \frac{\theta}{2} X_t(i)' \Sigma_{PD} X_t(i), \quad (9)$$

where Σ_{PD} represents the investor’s beliefs regarding the variance-covariance matrix of $P_{t+1} + D_{t+1}$.

As in most exponential-normal settings a linear price process can be found that will satisfy the standard conditions for a rational expectations equilibrium. In this case the appropriate conjecture is that

$$P_t = \frac{1}{r} D_t + B_0 + B N_t \quad (10)$$

where B_0 is a $k \times 1$ vector and B a $k \times k$ matrix. Using the conjectured price process (10) in (9) implies that investors will implicitly optimize

$$\max_{X_t} X_t' \left[BA(\bar{N} - N_t) + D_t - rP_t \right] + (1+r)W_t - \frac{\theta}{2} X_t' \left[\left(\frac{1+r}{r} \right)^2 \Sigma_\delta + B \Sigma_\eta B' \right] X_t \quad (11)$$

which yields the equilibrium conditions listed in the next proposition.

Proposition 1: In equilibrium

$$B_0 = \frac{1}{r} BA\bar{N}, \quad (12)$$

$$B = \frac{1}{2\theta} [rI + A] \Sigma_\eta^{-1/2} \left\{ -I + \left[I - 4\theta^2 \left(\frac{1+r}{r} \right)^2 \Sigma_\eta^{1/2} [rI + A]^{-1} \Sigma_\delta [rI + A]^{-1} \Sigma_\eta^{1/2} \right]^{1/2} \right\} \Sigma_\eta^{-1/2}, \quad (13)$$

and thus

$$P = \frac{1}{r} \left[D_t + B[rN_t + A\bar{N}] \right]. \quad (14)$$

Proof: See Appendix B for the proof of this and all other propositions.

Notice that the equilibrium equation describing B can always accommodate a singular dividend variance-covariance matrix (Σ_δ). In contrast, without a redefinition of the security set, the variance-covariance matrix of the supply process (Σ_η) must be of full rank.

While the B matrix appears rather complex one can simplify it to some degree. Define Γ as a matrix of orthonormal eigenvectors and Ω as a diagonal matrix of eigenvalues and write

$$\Gamma\Omega\Gamma^{-1} = 4\theta^2\left(\frac{1+r}{r}\right)^2 \Sigma_{\eta}^{1/2}[rI+A]^{-1}\Sigma_{\delta}[rI+A]^{-1}\Sigma_{\eta}^{1/2}, \quad (15)$$

which allows one to simplify B to

$$B = \frac{1}{2\theta}[rI+A]\Sigma_{\eta}^{-1/2}\Gamma[-I+(I-\Omega)^{1/2}]\Gamma^{-1}\Sigma_{\eta}^{-1/2}. \quad (16)$$

This brings out the importance of the square root term in B . The square root term produces two distinct equilibria for each eigenvalue since the i^{th} element in $(I-\Omega)^{1/2}$ has two roots $+\sqrt{1-\omega_i}$ and $-\sqrt{1-\omega_i}$, where ω_i represents the i^{th} diagonal element of Ω . Thus, if one can deal with portfolios that are proportional to the eigenvectors in Γ (eigen-portfolios) one can say a great deal about their behavior if one knows something about the associated eigenvalue. Note that within equation (16) the second term in square brackets contains what looks like a diagonal matrix of eigenvalues for B . While this is not exactly true, one can define a pseudo-eigenvalue matrix $\Lambda = .5[-I+(I-\Omega)^{1/2}]$ and its i^{th} diagonal element as λ_i . The next couple of propositions demonstrate that zero dividend portfolios are particularly easy to work with since they are eigen-portfolios, with associated eigenvalues in Ω equal to zero, and equal to either zero or minus one in Λ .

Empirical studies that examine closed-end mutual fund discounts are actually talking about the properties of a zero dividend portfolio that goes long the closed-end mutual fund and short the underlying assets. To single out such portfolios, let Z designate an arbitrary portfolio (with at least some non-zero security holdings) such that the holder of portfolio Z never expects to receive any net dividend payments. Thus, a portfolio that holds a short position in a closed-end fund and a long position in the underlying assets forms a special case within the set of all possible zero dividend portfolios.

The key to valuing a zero dividend portfolio lies within the following proposition.

Proposition 2: Let Z represent an arbitrary $k \times 1$ vector, while E and F represent arbitrary $k \times k$ matrices. Now assume that $Z' E = 0$. Then $Z' F [I - F^{-1} E F^{-1}]^{1/2} F' = \pm Z' F F'$.¹²

This proposition makes the analysis of a zero dividend portfolio tractable. In general, the matrix B in the equilibrium pricing equation (13) is very complicated. However, when pre-multiplied by a zero dividend portfolio and post-multiplied by $rI+A$ it collapses to the relatively simple $-(r\theta)^{-1} Z' [rI+A] \Sigma_\eta^{-1} [rI+A]$. This relationship allows one to say quite a bit about both the price and variance of the zero dividend portfolio.

Proposition 3: Assume Σ_δ is a singular matrix and that Z represents a portfolio such that $Z' D_t = 0$ in all states of nature, and that $Z' \Sigma_\delta = 0$. Then the value of the portfolio Z equals

$$Z' P_t = \begin{cases} -\frac{1}{r\theta} Z' [rI+A] \Sigma_\eta^{-1} [rN_t + A\bar{N}] & \text{if } \lambda_Z = -1 \\ 0 & \text{if } \lambda_Z = 0, \end{cases} \quad (17)$$

where λ_Z is the pseudo-eigenvalue of B associated with Z and

$$Z' \Sigma_p Z = \begin{cases} \frac{1}{\theta^2} Z' [rI+A] \Sigma_\eta^{-1} [rI+A] Z & \text{if } \lambda_Z = -1 \\ 0 & \text{if } \lambda_Z = 0 \end{cases} \quad (18)$$

equals its variance.

¹²Some care should be taken in the interpretation of Z . The portfolio Z specifies the number of asset units to be held within each firm. To get the actual shares of stock one needs to calculate $F_t = N_t^{-1} \cdot Z$ where N_t^{-1} represents the $k \times 1$ vector $(n_1^{-1}, n_2^{-1}, \dots, n_k^{-1})$. Thus, even if Z is time independent, the zero dividend portfolio will require rebalancing from period to period in response to the asset supply shocks. Fortunately, while these concerns require some care when attempting to implement the model empirically, they do not have much of an impact on the theoretical results or their interpretation.

For a zero dividend portfolio there exist two possible equilibria. When the associated eigenvalue equals 0 the standard set of results hold and the portfolio's value and variance both equal zero. However, there also exists another equilibrium (the negative root equilibrium) where the portfolio, despite paying zero in all states of nature exhibits a non-zero price and a positive variance. How can this be? The problem faced by potential arbitragers is that they may need to unwind their portfolio in a time period where the price has moved *even further* away from zero, thereby leading them to incur a loss. As in Dow and Gorton (1994), so long as agents have finite lives *they* do not view the "arbitrage" portfolio as providing *them* with an arbitrage since *they* may incur a loss if they try to take advantage of the situation. The pricing equation (17) tells us the amount by which the arbitrage portfolio's price can deviate from zero. Notice that it is not some arbitrary number. While agents are risk averse they are happy to take positions in portfolios that have sufficiently high returns given the risk faced by the individual purchasing the portfolio.

The economy's infinite horizon provides a key element within the model towards the creation of multiple equilibria. Without it, agents can backwards program the set of realized prices and the negative root equilibria will collapse. Thus, according to the model, a closed-end fund that everyone expects to "open up" should lose its discount.¹³ Brickley and Schallheim (1985) find something similar. While a fund's discount does eventually disappear when it announces plans to open up, the discount does not disappear all at once. One can reconcile this result with the current model if until the actual date of opening there remains some uncertainty as to whether or not the fund will in fact open up. Then the discount will narrow gradually as one approaches the day of reckoning at which point it will disappear. One can test this hypothesis by seeing if the discount narrows faster when funds provide greater assurances that they will in fact open up.

1.3 Equilibrium Selection and the Empirical Relevance of Multiple Equilibria

¹³A closed-end fund is said to "open up" if at some date it becomes an open ended fund.

While zero dividend portfolios that include closed-end funds may sell for a non-zero price, clearly there exist other zero dividend portfolio that do not. This duality of phenomena indicate that the economy can operate under more than one equilibrium, which in turn argues for a model with multiple equilibria. However, if there are multiple equilibria how does the economy select among them? One clue may be found in the data available to the public.

The data generated by a typical closed-end fund will lead to a noisy signal regarding the exact value of the discount. This noise enters the data through a number of avenues. For example, while funds alter their portfolio holdings nearly every day they only report its exact composition at periodic intervals. Thus, on a daily basis it will almost certainly be the case that a fund's market value will differ from its estimated net asset value based upon its most recently reported holdings. Furthermore, even if the fund's portfolio composition is known to the market transactions costs will still prevent arbitragers from equalizing the fund's price to its net asset value. Thus, when a trader looks at closed-end fund market data he will inevitably see a discount that moves over time. As a result, real world data will make it difficult to reject the hypothesis that any one fund is trading within the negative root equilibrium. Thus, given the available data a risk averse investor may decide to act as if the negative root equilibrium holds at least until there is some strong evidence to the contrary. To see why this is a prudent course, consider what a trader must do if he wishes to act as if he truly believes that the positive root equilibrium holds. In this case, he must take on the largest possible "arbitrage" position his creditors will allow (since he is acting as though he believes the position has a positive return and no risk). If by some chance the trader has guessed wrong about the equilibrium, he will then face possible financial ruin. Intuitively, one expects that most people will therefore "play it safe" by taking on partial positions. That is they will employ finitely elastic demand curves, which is what the negative root equilibrium requires to hold. Thus, if traders decide to act as if the negative root equilibrium holds (absent enough data to

reject this hypothesis), then in fact the negative root equilibrium will prevail.¹⁴

1.4 Closed-end Mutual Funds and Their Properties

So far the paper has only demonstrated that zero dividend portfolios may sell for a non-zero amount. For many applications, such as hedge fund management, this is sufficient. A hedge fund needs to know the likely price path of these portfolios so that they can gauge the risk-return tradeoff that they are facing. However, to explain the closed-end fund phenomenon one must go further. Closed-end fund values do not deviate from the value of their underlying assets by a mean zero amount. Rather they are generally priced at a discount, with occasional periods of time during which they sell at a premium. A model that tries to explain this phenomena should show more than the possibility that the closed-end fund will sell for a price that differs from the underlying securities. It should also explain why closed-end funds generally, but not always, sell for less. As noted earlier, this section of the paper will show, the current model can explain the above empirical regularities if one believes that the supply shocks associated with the mutual fund are small relative to those for the underlying stocks.

None of the variable definitions given so far distinguish a closed-end fund from any other corporation. However, if one is to account for the empirical regularities in the data it is clear that the model must somehow differentiate a closed-end fund from any other firm. As section 1.4 of the paper will show the key to explaining the data lies in the assumptions one makes about the variance of the capital supply shock associated with the closed-end fund. In particular, the model's predictions will conform to the known empirical regularities if one believes that the supply shocks associated with the mutual fund are small relative to those for the underlying stocks.

¹⁴I would like to thank Jeffrey Pontiff for the transactions costs argument. For a further discussion of the economic conditions under which the model's results will hold the reader should consult Appendix A. That appendix considers the economic forces that might lead the economy to choose one equilibrium over another, and whether or not the negative root equilibrium can be eliminated via the creation of additional long lived corporations.

Why, one might ask, should the capital supply for a closed-end fund should exhibit relatively little intertemporal variation when compared with the capital supplies of the underlying firms? Because, the supply shocks represent changes in corporate capacity and for a typical company (such as one that produces cars, steel, or cosmetics) the current supply of talent will have a fairly large impact on its aggregate output. Competent people can be expected to add greatly to the firm's capital base while incompetent ones will presumably subtract from it. Conversely, a closed-end mutual fund represents a financial institution that simply repackages claims. Changes to its capital base should be interpreted as changes in its ability to produce and mail statements, keep track of government paper work, and the other mundane tasks needed to operate the fund. No doubt these factors display some variation over time as the current talent pool changes. Nevertheless, it seems likely that this variation will be relatively small when compared to the capital shocks incurred by firms that produce real goods and services like cars and computers.¹⁵ Within the model this observation translates into a small variance for the supply shocks associated with the closed-end fund relative to the variance of the supply shocks for the other corporations in the economy. If the variance of a mutual fund's supply is in fact relatively small, then as section 1.4 will demonstrate the fund should sell at a discount, and that discount's variation should add to the fund's share price volatility.

To prove the following propositions it is useful to first define Y as the zero dividend portfolio that is short the mutual fund and long its underlying stocks, Y_u as the underlying stock

¹⁵If one derives the fund's dividend process from first principles, changes to its own capital base are the only relevant source of capital uncertainty. In period t the fund holds a position X_t in other firms going into the next period and thus produces a cash flow of $X_t D_{t+1} - G_{t+1}$. Since X_t is known in period t the only source of capital uncertainty lies in the G_{t+1} term and this varies with the productive efficiency of the fund. A formal derivation can be found at the end of Appendix B.

portfolio, and y_m as a portfolio that only holds the mutual fund.¹⁶ Thus,

$$Y = \begin{pmatrix} Y_u \\ \mathbf{0} \end{pmatrix} - \begin{pmatrix} \mathbf{0} \\ y_m \end{pmatrix} \quad (19)$$

and where the $\mathbf{0}$ vectors in (19) are included to make the terms in parenthesis conformable with each other.

Proposition 4: Assume that the variance of the mutual fund's supply shock ($\sigma_{\eta_m}^2$) is independent of all other supply shocks, and that A is a diagonal matrix. Further assume that the equilibrium in the economy sets pseudo-eigenvalue (λ_y) associated with Y equals -1 . Then, for $\sigma_{\eta_m}^2$ sufficiently small the expected value of the zero dividend portfolio Y is positive, and the variance of the closed-end fund's price will exceed the variance of its net asset value.

The results in Proposition 4 seem to conform with the two primary findings in the empirical closed-end fund literature. To a large degree closed end funds exhibit both relatively low prices and higher volatilities than a portfolio comprised of the underlying securities.

Notice that non-dividend shocks driving the results in Proposition 4 push the model in exactly the opposite direction from the analogous forces in the investor sentiment model of Lee, Shleifer, and Thaler (1991). In their model investor sentiment introduces a non-dividend based shock to the system, and they find that a higher variance on the investor sentiment parameter leads to a higher variance in the security. But in the current model, at least within the equilibrium of interest, a lower variance on the non-dividend based shock (i.e. the supply shock) leads to a higher variance and discount on the zero dividend portfolio containing the closed-end mutual fund. This difference may prove useful when attempting to reconcile the data. In general, theories have had

¹⁶Note that the portfolio that contains only the mutual fund is represented by a "y" in keeping with the convention of using lower case letters for scalar variables.

been unable to fully account for the rather large discounts observed in the data without the imposition of empirically unreasonable parameter values.¹⁷ However, in the current model a *small* variance on the mutual fund's supply shocks lead to large discounts. Thus, modest parameter values appear to have some chance of reconciling the data.

Another aspect of the equilibrium is the influence the elements of A have on both the existence of closed-end mutual fund discounts, and the volatility of the closed-end fund's per share price relative to that of the underlying assets. Once again consider the idea that human capital shifts influence the asset supplies. As argued earlier assets outside the closed-end fund represent companies that produce real goods and services and it seems likely that their production will depend heavily upon the current talent pool. In contrast, the talent pool for the production of closed-end fund services seems likely to be relatively stable. Within the model these relative influences are represented in part by the matrix A , with larger diagonal elements of A and smaller off diagonal elements of A representing a supply process that exhibits lower volatility over time as the talent pool changes.¹⁸ Thus, according to the arguments given above the diagonal value of A associated with the closed-end fund should be relatively large, and the off diagonal elements relatively small. As the next proposition shows, if these conclusions accurately represent the economic forces at work, then the closed-end fund will sell at a discount and exhibit excess price volatility relative to its net asset value.

¹⁷As discussed earlier Malkeil's (1977) paper came to this conclusion with regard to taxes and agency costs. More recently, a paper by Brauer (1993) comes to the same conclusion with regard to the investor sentiment hypothesis.

¹⁸The off diagonal elements have the opposite influence as the diagonal elements because they represent the influence of the supply of security i on the supply of security j . A more detailed explanation can be found in the discussion following Proposition 8.

Proposition 5: Assume that the variance of the mutual fund's supply shock ($\sigma_{\eta_m}^2$) is independent of all other supply shocks, and that A is a diagonal matrix. Let a_{mm} represent the element of A associated with the mutual fund. Then an increase in a_{mm} increases the closed-end fund's discount's expected value and variance.

The relationship documented in Proposition 5 provides a way to test the model against data outside that provided by the stock market. Actual aggregate corporate output equals the supply of outstanding shares times the current dividend produced by each share. Proposition 5 states that there should exist a relationship between the aggregate output statistics for the industries in a closed-end fund and that fund's discount. If the fund holds industries whose asset supplies mean revert relatively slowly then the discount on the fund will be relatively large.

Propositions 2 and 3 show that the zero dividend portfolio forms an eigenvector of both the dividend shocks' variance-covariance matrix and the square root term in B . Because this greatly simplifies the pricing equations one can say a great deal about the covariance of its price with any other portfolio.

Proposition 6: The covariance between a zero dividend portfolio and any other portfolio equals

$$\text{Cov}(Z'P, X'P) = \frac{1}{4\theta^2} Z' [rI+A] \Sigma_{\eta}^{-1} [rI+A] X. \quad (20)$$

Notice that even though the pricing equation is rather complex, the factors that influence the covariance between the zero dividend portfolio and any other portfolio are rather simple.

Some empirical studies have found that closed-end mutual fund discounts appear to decline when small firms have unusually large returns, and increase in periods where small firms exhibit below normal returns. This is really a statement about the correlation between closed-end fund

discounts and small stock returns, which are apparently correlated.¹⁹ Equation (20) shows how this can happen. Define $\Psi\Pi\Psi^I = [rI+A]\Sigma_n^{-1}[rI+A]$, so that Ψ equals a matrix of orthonormal eigenvectors, Π a diagonal matrix of eigenvalues with π_i as its i th diagonal element. With this definition one can decompose any arbitrary vector X into a weighted sum of the eigenvectors in Ψ . Thus, if there are k securities one can rewrite (20) as

$$\text{Cov}(Z'P, X'P) = \frac{1}{4\theta^2} \sum_{i=1}^k \pi_i w_i(Z) w_i(X) \quad (21)$$

where $w_i(Z)$, and $w_i(X)$ satisfy

$$Z = \sum_{i=1}^k w_i(Z) \psi_i, \text{ and } X = \sum_{i=1}^k w_i(X) \psi_i \quad (22)$$

if ψ_i equals the i th eigenvector in Ψ . Because, $[rI+A]\Sigma_n^{-1}[rI+A]$ is a positive definite matrix its eigenvalues must be strictly positive. Equation (21) basically describes a principal components analysis of the correlation between the zero dividend portfolio and any other portfolio. The empirical statement that closed-end fund discounts appear correlated with the returns to small company stocks is equivalent (within the model) to a statement that they share similar weights (w)

¹⁹There is quite a bit of disagreement about this conclusion. Lee, Shleifer, and Thaler (1991) presented the initial evidence regarding the correlation between closed-end fund discounts and small stock returns. Chen, Kan and Miller (1993a, and 1993b) then presented new evidence that this correlation either does not exist or it is very weak. They also reinterpret the findings in Lee, Shleifer and Thaler (1991) to imply that the evidence does not support the existence of a correlation between closed-end fund discounts and small stock returns. Chopra, Lee, Shleifer, and Thaler (1993a, 1993b) dispute their conclusions. Whatever the case may be, the predictions presented in this paper are about the potential correlation between closed-end fund discounts and any other arbitrary portfolio. While one can apply the model's findings to small firm portfolios, it is also possible to apply them to portfolios with other characteristics.

in (22).²⁰ Notice, that the equation underlying the principal components decomposition (equation (20)) does not include the variance-covariance matrix governing the dividend process. This is not too surprising since the variance of a zero dividend portfolio obviously equals zero. However, it does mean that the covariance between the zero dividend portfolio and the underlying securities depends on variance-covariance matrix of the supply shocks. The rather extensive literature on small stock returns indicates that small stocks incorporate a return “factor” that does not appear to influence large company stocks. Thus, there is every reason to believe that one of the eigenvectors in Ψ may represent a “small stock” portfolio. Under this scenario the closed-end fund discount will be correlated with small stock returns if the appropriate value of $w(Z)$ is positive.

Another empirical phenomena revolves around the finding by Thompson (1978) and Pontiff (1995) that closed-end fund discounts tend to be mean reverting. As the next proposition shows this empirical pattern can be guaranteed within the model by simply adding the assumption that A is a diagonal matrix equal to aI , where $a > 0$ is a scalar.

Proposition 7: If A can be written as aI then under the negative root equilibrium the closed-end fund discount will exhibit mean reversion. Mathematically, let $Y \hat{P}(\bar{N})$ represent the expected value of the closed-end fund discount. If $Y \hat{P}_t > Y \hat{P}(\bar{N})$ then $E(Y \hat{P}_{t+1}) < Y \hat{P}_t$, and if $Y \hat{P}_t < Y \hat{P}(\bar{N})$ then $E(Y \hat{P}_{t+1}) > Y \hat{P}_t$.

Clearly one can greatly relax the proposition’s restrictions on A and still retain the basic result in a number of cases. Intuitively, if the underlying production process in the economy exhibits mean reversion, then so will various statistics produced by the stock market.

²⁰This type of analysis resembles some of the statistical work found in Lee, Shleifer and Thaler (1991) and Chen, Kan and Miller (1993a). These papers regress closed-end fund discounts on the returns for 10 different stock size deciles. One can think of this as a search for the eigen-portfolio with the largest eigenvalue.

The above propositions outline the parameter values that lead to expected closed-end fund discounts. Nevertheless, with some probability the discount will temporarily turn into a premium if the realized supply shocks take on the appropriate values. Thus, the model offers a potential explanation for the creation of closed-end funds, and their subsequent returns. In those periods of time when there exists a premium investment firms can profitably create closed-end funds.²¹ Then, due to the mean reversion in the discount (Proposition 7), the premium should then proceed to disappear and eventually become a discount.

At first blush, the above results seem to conform with Peavy's (1990) empirical findings on the creation of new closed-end funds, their initial prices and subsequent returns. Also, one should note, that the very same patterns arise in the investor sentiment model of Lee, Shleifer, and Thaler (1991). Alas, neither the current explanation nor that provided by the investment sentiment model is perfect. While it is true that closed-end funds are initially issued at a premium, Weiss-Hanley, Lee and Seguin (1996) show that the price would immediately collapse in the aftermarket without the very strong support provided by the investment firms. If in fact closed-end funds are only issued when they should sell at a premium then this type of support should not be needed. Thus, while both this model and the investor sentiment model can account for the creation of closed-end funds, they cannot reconcile all of the data. It thus seems more likely that the economic forces leading to the creation and sale of new closed-end funds differ from those that determine prices in the secondary market.

1.5 The Relationship Between Closed-End Fund Discounts and Aggregate Production

Intuitively, the most direct relationship between closed-end fund discounts and corporate

²¹Obviously, within the current model investment houses cannot expend resources to create new funds. While ideally one would like to add this feature explicitly, the mathematics become considerably more complicated and would take the general discussion far afield. Nevertheless, one suspects that most of the primary results would not change.

productivity should involve changes in the contemporaneous values. Indeed there seems to be some indirect evidence that this relationship is a positive one. That is, when the economy does well closed-end fund discounts seem to increase. The evidence is only indirect since most studies have only examined the relationship between closed-end fund discounts and stock market returns, and this is not the same thing as current corporate productivity. However, with that caveat in mind, both Malkiel (1977) and Pontiff (1995) find that when the stock market does well, closed-end fund discounts increase. Thus, one might expect a study that focused on corporate output to find a similar relationship, as would be predicted by the next proposition.

Proposition 8: The covariance between changes in the discount, and changes in total current corporate production equals

$$\text{Cov}(Y'(P_{t+1}-P_t), N'_{t+1}D_{t+1}-N'_tD_t) = -\frac{1}{\theta}Y'AD_t. \quad (23)$$

Let the subscript m designate the location in the vector Y that holds the short position in the closed-end mutual fund. Then the covariance increases as a_{mm} increases, and if all of the values of Y_w and D_t are non-negative, then the covariance increases as a_{im} decreases.

The proposition shows two things. First, decreasing the values of the off diagonal elements of A associated with the closed-end fund increases the covariance. Intuitively, one can think of these elements as determining the influence that asset supplies in one security have on the asset supplies of another. When the a_{ij} ($j \neq i$) are negative then large supplies in one asset tend to drive down supplies in other assets. Second, for the diagonal element of A associated with the closed-end fund, increasing its value increases the covariance. As discussed earlier increasing this parameter increases the speed of mean reversion in the asset's supply. Together these two points lead to a simple inference. If asset supplies mean revert fast enough then closed-end fund discounts will be

positively correlated with stock prices. Cross-sectionally the implication is that an increase in the speed at which supplies mean revert leads to an increase in the covariance between changes in the discount and total current corporate production.²² This can be tested by looking at the rate of mean reversion in the productive capacity of firms across closed-end fund portfolios. Those portfolios holding firms with the greatest degree of mean reversion should have the highest correlation between their discount and market prices.

Proposition 8 deals with contemporaneous changes in the closed-end fund discount and corporate production. However, based on Swaminathan's (1996) study, it also appears that there exists a relationship between closed-end funds discounts and *future* corporate output.

Unfortunately, fitting the model to Swaminathan's (1996) results runs into problems associated with the "economy's initial conditions." The mean reverting supply process prevents one from simply calculating a meaningful relationship between closed-end fund discounts, and aggregate production. Not only will the results depend upon the initial starting conditions, but also the one, two, three and other future correlations will differ from each other. Another problem lies with the fact that empirical studies do not provide the necessary information to calculate a set of reasonable starting conditions for the security supplies. All of this suggests that it may prove more fruitful to look for correlation statistics that do not depend upon the model's initial conditions, at least with respect to the supply process.²³

²²One can increase the speed of mean reversion either directly, via an increase in the a_{ii} , or indirectly via a decrease in the a_{ij} for $j \neq i$.

²³As will become apparent, to a limited degree, the results in this section do depend upon the initial starting conditions for the dividend process. There is nothing that can be done about this since the model assumes that the dividend process does not mean revert. Thus, the expected value of the dividend vector in any future period is simply its current value. One can eliminate this problem by simply imposing a mean reverting process on the dividends. That will only change the results in the subsequent sections in that a vector representing the long run average dividend (a term analogous to \bar{N}) would replace D_0 in the equations that follow.

In order to calculate unconditional statistics, one needs to examine the model's "steady state." This can be achieved by looking at the period t value of the statistic based upon the available information at time 0, and then letting t go to infinity. In this way the influence of the initial conditions can be purged from the results. For the problem at hand, the unconditional covariance between the closed-end fund's discount and future corporate productivity equals:

$$\lim_{t \rightarrow \infty} \text{cov}(Y' P_p [N_{t+1}' D_{t+1} - N_t' D_t] | N_0) \equiv \lim_{t \rightarrow \infty} E_0 \left\{ Y' [P_t - P(\bar{N})] [N_{t+1}' D_{t+1} - N_t' D_t] \right\} \quad (24)$$

where $P(\bar{N})$ represents the price vector given N equals \bar{N} , and N_0 represents the value of N in period 0. The first term in the covariance calculation equals the period t discount on the closed-end mutual fund relative to its unconditional mean. The second term equals the difference between aggregate corporate production in periods $t+1$ and t .

Proposition 9: The unconditional covariance between closed-end fund discounts and future aggregate production equals

$$\lim_{t \rightarrow \infty} \text{cov}(Y' P_p [N_{t+1}' D_{t+1} - N_t' D_t] | N_0) = \frac{1}{\theta} Y' [rI + A] \Sigma_\eta^{-1} \left[\sum_{\tau=1}^{\infty} (I - A)^{\tau-1} \Sigma_\eta (I - A)^{\tau-1} \right] A D_0 \quad (25)$$

Proposition 9 provides an empirical prediction regarding current discounts and future aggregate corporate production; a result that cannot be readily produced by models that rely on either investor sentiment or transactions costs. While investor sentiment may be correlated with current market conditions, why should it forecast future aggregate production? Transactions costs models suffer from similar problems. While they may be associated with current economic predictions it is not clear why they should forecast future conditions.

To obtain additional insights into Proposition 9's results consider the case where Σ_η and A

are both diagonal matrices. In this case the right hand side of equation (25) simplifies to

$$\text{cov}(Y' [P_t - P(\bar{N})], [N_{t+1}' D_{t+1} - N_t' D_t]) = \sum_{i=1}^k \frac{1}{\theta} y_i d_{0i} \frac{r + a_{ii}}{2 - a_{ii}}. \quad (26)$$

Again, in line with the notation developed so far, small letters stand for scalar elements within their corresponding matrices. Thus, y_i equals the i th element of Y , while d_{0i} represents the period 0 dividend provided by security i , and a_{ii} equals the i th diagonal element of A . Recall, that Y represents a zero dividend portfolio that holds a long position in the closed-end fund's underlying assets and a negative position in the closed-end fund itself. Consider the case where all of the initial dividends are positive and the closed-end mutual fund only holds long positions in its underlying assets. Then the i th element in the summation within equation (26) will take on non-negative values for securities held by the closed end fund (since the corresponding y_i will be positive in such cases) and a non-positive value for the element of y_i representing the short position in the closed-end fund itself.

The actual magnitude of (26) clearly depends upon a number of elements. However, the influence of each individual a_{ii} , the rate at which a particular supply process mean reverts, can be broken out to produce some empirically interesting properties.

Proposition 10: Assume that Σ_η and A are diagonal matrices. Then an increase in a_{mm} or a decrease in a_{ii} for $i \neq m$ decreases the unconditional covariance between the current closed-end fund discount and future corporate production.

Once again, a relatively small valued mean reversion parameter for the closed-end fund's supply leads to results that are roughly consistent with the data, in this case a negative correlation between the discount and future economic activity.

By now it should be clear that the model can reconcile a number of empirical patterns via

the same restrictions on the relative speed with which asset supplies mean revert. Proposition 5 shows that an increase in a_{mm} (the element of A associated with the closed-end fund) leads to a larger discount and higher volatility on closed-end fund's price. Proposition 8 shows that increasing the value of a_{mm} increases the covariance between the discount and changes in current corporate productivity. Finally, Proposition 10 shows that increasing a_{mm} induces a negative covariance between current closed-end fund discounts and future production. Thus, the model provides a fairly parsimonious explanation for a number of observed phenomena.

Together Propositions 5, 8, and 10 link the same set of parameter values to multiple results, and thus they provide a set of associated testable hypotheses. If the underlying industries held by a closed-end fund show relatively slow rates of mean reverting production then one expects this to imply four things about the closed-end fund's price:

1. it will display a relatively large discount,
2. it will be relatively volatile,
3. it will have a particularly strong covariance with unexpected contemporaneous changes in corporate valuations, and
4. it will display a negative covariance with future production.

Furthermore, these hypotheses are easily tested against the empirical data. Simply look at the industry holdings for each fund, and then compare the rate at which productivity within each industry mean reverts relative to its trend. Funds that hold industries with relatively slow levels of mean reversion should be responsible for a disproportionate fraction of the statistical patterns that have been observed in the data between close-end fund discounts and the other economic variables that researchers have examined.

2 Relation to the Theoretical Literature

Lee, Shleifer and Thaler (1991) argue that investor sentiment drives the observed patterns associated with closed-end fund discounts. In their model the economy contains both rational investors and those influenced by “sentiment.” They argue that if investor sentiment is correlated across individuals then their demands can move prices if rational agents are unwilling to take fully offsetting positions.

Within the current model the random capital stock plays a very important role and one that appears analogous to investor sentiment in Lee, Shleifer and Thaler (1991). Thus, the reader might legitimately ask if there are any equilibrium features that distinguish the two. In fact, there are at least four major differences. First, because all of the agents within the current model maximize utility functions there always exists an equilibrium in which closed-end fund discounts do not exist. Thus, even with random capital supplies closed-end fund discounts do not have to exist. Depending upon how one sets up a noise trader or sentiment model this may or may not be true.²⁴ Second, within equilibria that permit closed-end fund prices to deviate from their net asset value, a *decrease* in the uncertainty regarding the fund’s future production stock for the closed-end fund *increases* the discount and the volatility of the fund’s per share price. Conversely, in a model of investor sentiment such as Lee, Shleifer, Thaler (1991), an increase in the demand volatility of such traders increases the fund’s price volatility as well as the discount.

A third difference between an investor sentiment model and one using a mean reverting supply process lies in how they tie the empirical phenomena that closed-end fund discounts exhibit negative intertemporal serial correlation to other characteristics of the economy. In a random supply model changes in the discount are tied directly to changes in corporate assets and this ties the rate of mean reversion to concurrent changes in aggregate production. Fourth, tying the

²⁴As Bhushan, Brown and Mello (1997) demonstrate many results within the investor sentiment literature depend critically upon the assumed form of irrational behavior.

discount on closed-end funds to corporate productivity also induces a relationship between the current discount and future changes in corporate output. As the analysis of the model shows the parameter values that lead to the existence of closed-end fund discounts also lead to a positive correlation between the discount and concurrent changes in value of the overall corporate productivity. Simultaneously, these same parameter values also lead to a negative correlation between closed-end fund discounts and future corporate productivity. In contrast, it is not clear why models of investor sentiment should produce non-zero correlations among current discounts and future production levels.²⁵

Another model that examines the closed-end discount can be found in Chordia and Swaminathan (1997). They propose an explanation that includes both asymmetric information and market segmentation. In their model market segmentation occurs because some of the investors are restricted from trading in some markets. They then show that if there are two identical securities, and if the informed investors cannot trade in one of them, then the other security (the “closed-end fund”) may sell at a discount. This type of segmentation, they argue, is consistent with evidence in Weiss (1989) that institutional investors tend to shun closed-end funds.²⁶

3 Conclusion

Financial theory rests to a large degree upon a paradigm in which stocks trade within a

²⁵This point can also be found in Swaminathan (1996).

²⁶The actual implications one should draw from Weiss’ (1989) findings seem to be in dispute. Lee, Shleifer, and Thaler (1991) argue that small investors are more likely to herd via correlated investor sentiment than are large institutional money managers. Thus, they take the lack of institutional money in closed-end funds to imply that investor sentiment will play a larger role in the price process. However, there also seems to be some dispute as to which investors, if any, are more prone to herding. Froot, Scharfstein and Stein (1992) conjecture that professional money managers are the ones likely to herd. In terms of the direct empirical work on this issue, Grinblatt, Titman, and Wermers (1995) find some evidence of herding by mutual fund managers, but do not show if there is more or less relative to herding among small investors.

frictionless market, populated by competitive agents possessing homogenous information sets. Even absent this, at the very least most theories rule out arbitrage possibilities. Alas, this world view cannot explain why the stock market frequently prices closed-end mutual funds at a substantial discount from their net asset values. In a frictionless market arbitrage, if nothing else, should preclude any discrepancies from occurring let alone long term discrepancies with non-zero means.

As Malkiel (1977) points out, the standard financial paradigm (even with the addition of some standard frictions such as taxes and agency costs) simply cannot explain the large persistent closed-end fund discounts observed in the data. However, the model presented here shows that one can reconcile large parts of the data by altering the standard financial model in other modest ways. In particular if asset supplies vary over time, and if rational investors have finite lives then closed-end fund discounts can appear in what is otherwise a very traditional model. These discounts appear and persist despite a lack of frictions from either transactions costs, or informational asymmetries. For better or worse it appears that death is a rather large friction. Individuals with finite lives cannot arbitrage between closed-end funds and the underlying securities simply because they risk having the price move against them during their lifetime. If these investors believe that closed-end fund prices can deviate from the fund's net asset value, then they will act accordingly by submitting demands that exhibit some inelasticity. Such behavior will of course verify their beliefs since closed-end fund prices will then deviate from underlying net asset values.

The model not only provides a setting in which closed-end fund discounts can exist, but it also produces a number of empirical predictions that also seem to concur well with patterns in the data. In particular, a number of propositions show that the same parameter values that lead to the existence of a closed-end fund discount will also lead to a fund price that is more volatile than the underlying net asset value, to a discount that exhibits a positive covariance with current changes in

corporate production, and whose magnitude displays a negative covariance with future economic productivity. Since all four of these regularities appear in the data, the model may provide a simple explanation capable of connecting them to one source.

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5 Appendix A - Stability of the Equilibria

Both roots of the eigenvalue matrix yield equilibria that are fully consistent with the generally accepted principles governing Nash equilibria. In all cases each trader's actions maximize his or her utility given the actions of all other traders, while the equilibrium prices equilibrate supply and demand in each asset. Thus, issues of stability (to the extent they exist at all) must be addressed outside the usual game theoretic arguments.

One way to examine the stability of each equilibrium is to see whether the system tends to “drift towards or away” from the equilibrium point if it starts a small distance away from the equilibrium. Such arguments are necessarily heuristic since game theoretic arguments do not include an explanation as to how the economy selects one equilibrium or another. However, with this caveat in mind the forces that drive the two equilibria can be seen within the following graph.

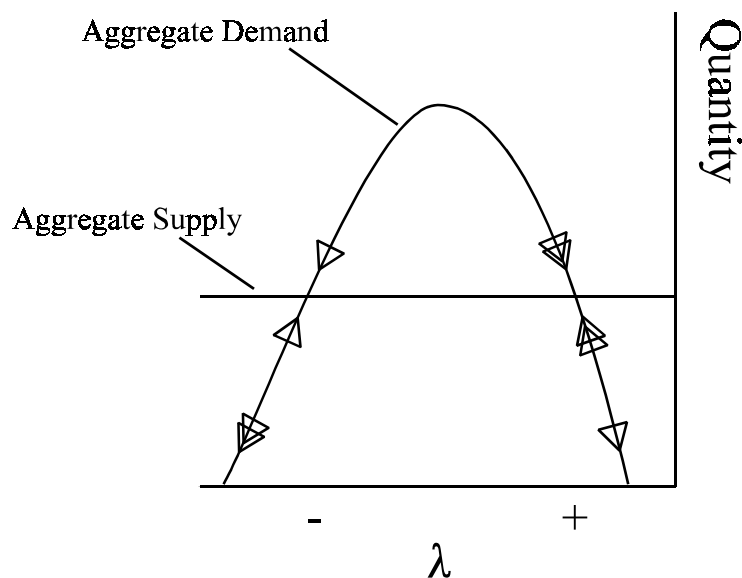


Figure 1: “-” and “+” = value of some eigenvalue λ under the negative and positive root equilibria respectively. Single triangles represent the “out of equilibrium” forces on λ from the stock’s variance, while the double triangles do the same for the stock’s price.

The arc in Figure 1 represents the aggregate demand of the population for the eigen-portfolio associated with the pseudo-eigenvalue λ , while the straight line represents the eigen-portfolio's aggregate supply. In the scalar case the value of B is simply proportional to λ , thus in the vector case one can think of an increase in λ as equaling an increase in B . In fact, this interpretation will make the following discussion somewhat easier to understand in terms of the influence the pseudo-eigenvalues have on stock price volatility via equation (14).

The plus and minus signs within Figure 1 designate the value of λ under the positive and negative root equilibria. Notice, that there are two market clearing values for λ because an increase in λ does not uniformly increase or decrease the aggregate demand. With a positive supply of the asset, an increase in λ has two opposing influences on demand. First, it increases the stock price thereby discouraging investors from purchasing the asset. Second, it reduces the variance of the stock's price (since an increase in λ moves its value towards zero) which encourages investors to purchase the asset. Depending on the current value of λ one or the other influence will dominate.

The arrowheads in Figure 1 indicate the direction of each force's influence on an auctioneer that uses a tatonnement process to set λ by considering λ 's influence on the price and variance separately. First, consider the area to the right of the plus sign. In this region, the auctioneer notices that there exists insufficient demand for the stock. The auctioneer can increase demand by lowering the price variance, and he can do this by increasing λ . Thus, the single arrowhead points to the right. Alternatively, the auctioneer can increase demand by decreasing the price of the stock, which he can accomplish by decreasing λ . Thus, the double arrowheads point to the left and into the plus sign. These same arguments can also be applied to the other regions in the graph thereby yielding the forces indicated by the remaining arrowheads.

Notice that in terms of the stability arguments given above neither the positive nor the negative root equilibrium has an advantage. The impact of λ on the security's price tends to favor the positive root equilibrium, while its impact on the variance tends to favor the negative root equilibrium.

One might suspect that even if an equilibrium is stable for an economy with finitely lived investors, it may not be stable if these same investors can set up infinitely long lived mutual funds. In order to make the argument concrete, consider a period t generation that attempts to profit by setting up such a mutual fund when the price of the zero dividend portfolio has a positive price. Under this scenario the fund sells the zero dividend portfolio and distributes the cash to its investors. In period $t+1$ the investors then offer to sell their shares in the fund to next generation at a price of zero by pointing out that there are no cash flows associated with the fund. As part of the old generation's sales pitch they must convince the young generation that too will find ready buyers at a price of zero one period later. In fact, the period $t+1$ old generation must convince the young generation that every subsequent generations will successfully convince its own descendants to accept the fund shares at a price of zero.

Now, suppose that after setting up the fund the price of the zero dividend portfolio increases even further, and consider the position of the generation that is "supposed" to purchase the shares at a price of zero. If these investors want to hold this same mutual fund they can go out and sell the same zero dividend portfolio themselves, and then *keep* the revenues themselves. Clearly, this strategy at least weakly dominates that of purchasing the fund shares from the previous generation. The young generation winds up holding the same assets plus the cash from shorting the underlying securities, rather than with a position in just the underlying securities. Even so, might the young generation still agree to make the purchase? Not if they think there exists an arbitrarily small possibility that the next generation will not agree to transfer the shares

at a level below the fund's net asset value. Consider such "trembles" and let the trembles go to zero. For any positive tremble each member of the young generation strictly prefers to create his or her own position by trading in the underlying assets rather than accept the mutual fund shares from the old generation. Thus, under what would seem to be a fairly reasonable application of "trembling hand" perfection, the creation of an infinitely lived mutual fund cannot eliminate the negative root equilibrium. The mutual fund is simply a corporate shell and cannot alter the fundamental tradeoffs available to investors.²⁷ Unfortunately, no corporation or mutual fund can change the fact that the underlying claimants have finite lives.

²⁷As with any other model in this genre, all of the shocks are normally distributed in order to make the model tractable. Thus, prices can go negative and investors do not enjoy limited liability protections. However, the dominance argument does not appear sensitive in this regard and if anything limited liability should strengthen them. With limited liability a mutual fund with a negative net asset value will simply declare bankruptcy. This possibility may severely limit its ability to short the "expensive" security.

6 Appendix B

Proof of Proposition 1: Given the wealth equation (5) and the conjectured pricing equation (10) the trader will seek to solve the problem given by equation (11). Differentiating (11) therefore produces the first order conditions that must be satisfied by the investor's demands. Next, match terms to solve for B_0 , and B . Since aggregate demand cannot depend upon the current dividend, one must have that

$$B_0 = \frac{1}{r} B A \bar{N}. \quad (27)$$

Now set the integral of the $X_i(i)$ equal to N_t in order to produce an equation in terms of the matrix B ,²⁸

$$B \Sigma_{\eta} B' + \frac{1}{\theta} B [rI + A] + \left(\frac{1+r}{r} \right)^2 \Sigma_{\delta} = 0. \quad (28)$$

This is a quadratic matrix equation in B . To solve it let $B_t = B[rI + A]$, and use this to replace B in the above equation. After making this substitution pre and post multiply the equation by $\Sigma_{\eta}^{1/2} [rI + A]^{-1}$ and its transpose respectively. Now let

²⁸While switching the control variable from $F_t(i)$ to $X_t(i)$ does not alter the model's equilibrium properties it does require a slight alteration in the equations that must be satisfied in equilibrium. Having normalized the number of outstanding shares in each firm to one, equilibrium normally requires that

$$\int F_t(i) di = 1$$

which is simply that demand equals supply. However, with the change from $F_t(i)$ to $X_t(i)$ this condition now becomes the equivalent

$$\int X_t(i) di = N_t.$$

Otherwise, everything else remains unchanged.

$$Y = \Sigma_{\eta}^{1/2} [rI+A]^{-1} B_1 [rI+A]^{-1} \Sigma_{\eta}^{1/2} \quad (29)$$

and note that Y is a symmetric matrix, implying that $Y = Y'$. With all of these substitutions in place the quadratic equation reduces to

$$Y^2 + \frac{1}{\theta} Y + \left(\frac{1+r}{r} \right)^2 \Sigma_{\eta}^{1/2} [rI+A]^{-1} \Sigma_{\delta} [rI+A]^{-1} \Sigma_{\eta}^{1/2} = 0. \quad (30)$$

In order to solve (30) for Y add and subtract $.25\theta^2 I$, which will complete the square. This yields a solution for Y equal to

$$Y = -\frac{1}{2\theta} I + \left[\frac{1}{4\theta^2} I - \left(\frac{1+r}{r} \right)^2 \Sigma_{\eta}^{1/2} [rI+A]^{-1} \Sigma_{\delta} [rI+A]^{-1} \Sigma_{\eta}^{1/2} \right]^{1/2}. \quad (31)$$

Reversing through all of the substitutions produces equations (12), (13), and (14). Q.E.D.

Proof of Proposition 2: Proof: Let S and V represent $k \times k$ matrices, with V diagonal, such that $SVS^{-1} = F^{-1}EF'^{-1}$. Thus, S represents a matrix of eigenvectors and V a diagonal matrix of eigenvalues for $F^{-1}EF'^{-1}$. Using these relationships the following equalities must hold,

$$\begin{aligned} Z'F[I-F^{-1}EF'^{-1}]^{1/2}F' &= Z'F[I-SVS^{-1}]^{1/2}F' = \\ Z'F[S(I-V)S^{-1}]^{1/2}F' &= Z'FS(I-V)^{1/2}S^{-1}F. \end{aligned} \quad (32)$$

From the definitions of F , S , and V the following equalities must also hold

$$Z'FSVS^{-1}F' = Z'FF^{-1}EF'^{-1}F' = Z'E = 0, \quad (33)$$

where the last equality follows from the proposition's assumption that $Z'E=0$. Thus, Z is an eigenvector of $F^{-1}EF'^{-1}$ with an associated eigenvalue 0. Define $Z'F = Y'$ and select S orthonormal

with the exception of the eigenvector that is proportional to Z . Call the eigenvector associated with Z , s_z and normalize it so that $Z's_z = 1$. Eigenvectors with these properties must exist since $F^{-1}EF'^{-1}$ is symmetric, see Strang (1976). Now the i^{th} diagonal element of $(I-V)^{1/2}$ equals ± 1 since the i^{th} diagonal element of V equals 0. So,

$$Z'FS(I-V)^{1/2}S^{-1}F' = \pm Z'FF' \quad (34)$$

since the square root of +1 equals ± 1 . Q.E.D.

Proof of Proposition 3: To derive equation (17) pre-multiply equation (14) by Z' after using (13) to substitute out for B . Then use Proposition 2 to produce (17).

To derive (18) note that

$$\Sigma_P = \frac{1}{r^2}[\Sigma_\delta + B\Sigma_\eta B'] \quad (35)$$

Now pre and post multiply by Z and then use Proposition 2 to derive (18). Q.E.D.

Proof of Proposition 4: Under the negative root equilibrium, the expected value of the zero dividend portfolio Y can be written as

$$E(Y'P) = -\frac{1}{r\theta}Y'[rI+A]\begin{bmatrix} \Sigma_{\eta 1}^{-1} & 0 \\ 0 & \sigma_{\eta m}^{-2} \end{bmatrix}[rI+A]\bar{N}. \quad (36)$$

Here $\Sigma_{\eta 1}$ represents the supply shock variance-covariance matrix for the securities in Y_1 .

Expanding equation (36) produces

$$E(Y'P) = -\frac{1}{r\theta}\left[Y_1'(rI_1+A_1)\Sigma_{\eta 1}^{-1}(rI_1+A_1)\bar{N}_1 - y_m(r+a_m)^2\sigma_{\eta m}^{-2}\bar{n}_m\right] \quad (37)$$

where variables with the subscript “1” represent the appropriate submatrices and subvectors that correspond to the elements in Y_1 . The variables with a subscript “m” represent the scalar elements of the appropriate matrices and vectors associated with y_m . Clearly for $\sigma_{\eta m}^2$ small enough $E(Y'P)$ must be strictly positive, proving the first claim in the proposition.

Under the negative root equilibrium the payoff variance from holding Y equals

$$Y' \Sigma_p Y = \frac{1}{\theta^2} Y' [rI + A] \begin{bmatrix} \Sigma_{\eta 1}^{-1} & 0 \\ 0 & \sigma_{\eta m}^{-2} \end{bmatrix} [rI + A] Y. \quad (38)$$

Expanding this equation into the components associated with Y_1 and y_m produces an equation similar to (37) with the \bar{N} and \bar{n} terms replaced by Y_1 and y_m respectively. Taking the limit as $\sigma_{\eta m}^2$ goes to zero proves the second claim in the proposition. Q.E.D.

Proof of Proposition 5: The proof of the first statement in the proposition follows immediately from equation (36). The proof of the second statement uses (38) and follows along the same lines as the arguments given in Proposition 4 following (38). Q.E.D.

Proof of Proposition 6: The derivation is similar to that of equation (18). Use the same arguments with Z as the zero dividend portfolio, and X as an arbitrary portfolio. Q.E.D.

Proof of Proposition 7: Using (17) write the realized change in the closed end fund discount as,

$$Y'(P_{t+1} - P_t) = -\frac{1}{\theta}(r+a)Y' \Sigma_{\eta}^{-1}(N_{t+1} - N_t). \quad (39)$$

now use (1) to find the expectation of (39) as

$$E[Y'(P_{t+1} - P_t)] = -\frac{1}{\theta}a(r+a)Y' \Sigma_{\eta}^{-1}(\bar{N} - N_t). \quad (40)$$

The difference between the current closed end fund discount and its unconditional expected value equals

$$Y'(P_t - P(\bar{N})) = -\frac{1}{\theta}(r+a)Y'\Sigma_\eta^{-1}(N_t - \bar{N}), \quad (41)$$

which clearly has the opposite sign as (40) since $0 \leq a \leq 1$. Q.E.D.

Proof of Proposition 8: From equation (17) $Y'(P_t - P_{t+1})$ can be written as

$$Y'(P_t - P_{t+1}) = -\frac{1}{\theta}Y'[rI+A]\Sigma_\eta^{-1}(A(\bar{N} - N_t) + \eta_{t+1}). \quad (42)$$

From the equations specifying the dividend and supply processes (equations (2) and (1)) one finds that

$$N'_{t+1}D_{t+1} - N'_t D_t = (A(\bar{N} - N_t) + N_t + \eta_{t+1})'(D_t + \delta_{t+1}) - N'_t D_t \quad (43)$$

Plugging the above two equations into the formula for the covariance between two variables produces

$$E\left\{-\frac{1}{\theta}Y'[rI+A]\Sigma_\eta^{-1}\eta_{t+1}\eta'_{t+1}D_t\right\} = -\frac{1}{\theta}Y'[rI+A]D_t = -\frac{1}{\theta}Y'AD_r \quad (44)$$

where the second equality follows from the assumption that $Y'D_t = 0$. Q.E.D.

Proof of Proposition 9: To prove the proposition use equation (17) to write

$$Y'(P_t - P(\bar{N})) = -\frac{1}{\theta}Y'[rI+A]\Sigma_\eta^{-1}(N_t - \bar{N}). \quad (45)$$

Next, simplify the second term in the covariance calculation by writing the period $t+1$ values for the dividend and security supply in terms of period t values and shocks. Thus,

$$N_{t+1}'D_{t+1} - N_t'D_t = [A(\bar{N} - N_t) + \eta_{t+1}]'D_t + [A(\bar{N} - N_t) + N_t + \eta_{t+1}]'\delta_{t+1}. \quad (46)$$

Now plug equations (45), and (46) into the covariance formula in (25), and then eliminate the D_t by using

$$D_t = D_0 + \sum_{\tau=1}^t \delta_{\tau}. \quad (47)$$

At this point, note that the N_t terms do not depend upon the realization of any of the δ_t terms and recall that the δ_t have both mean zero expectations and zero correlations with any other non-dividend shocks within the system. Thus, the expectation of δ_t with either N_t or η_t for any t or τ equals zero. This eliminates all of the δ_t terms from the calculation.

To eliminate the N_t terms one needs to calculate the value of N_t in terms of N_0 . To do this repeatedly use (1) to move forward in time from period 0 to period t and thereby produce

$$N_t = (I-A)^t N_0 + \sum_{\tau=1}^t (I-A)^{\tau-1} [A\bar{N} + \eta_{t+1-\tau}]. \quad (48)$$

The as t goes to infinity $(I-A)^t$ goes to zero and $\sum_{\tau=1}^t (I-A)^{\tau-1} A\bar{N}$ goes to \bar{N} since (by assumption) the eigenvalues of $I-A$ lie between zero and one. Thus, both the N_0 and \bar{N} terms disappear from the calculation of (25). This leaves the following elements to account for in the limit

$$\lim_{t \rightarrow \infty} \text{cov}(Y' [P_t - P(\bar{N})], [N_{t+1}' D_{t+1} - N_t' D_t]) =$$

$$\lim_{t \rightarrow \infty} E \left\{ \frac{1}{\theta} Y' [rI + A] \Sigma_{\eta}^{-1} \left(\sum_{\tau=1}^t (I-A)^{\tau-1} \eta_{t+1-\tau} \right) \left(\sum_{\tau=1}^t (I-A)^{\tau-1} \eta_{t+1-\tau} \right)' A - \eta'_{t+1} \right\} D_0 \quad (49)$$

the right hand side of which simplifies to (25). Q.E.D.

Proof of Proposition 10: Simply differentiate equation (26) to prove the proposition. Q.E.D.

Derivation of the claim in footnote 15: Footnote 15 states that only the volatility concerning the closed-end fund's own production capital influences the discount. To prove this begin by

assuming, without loss of generality, that the closed-end fund is the first security in the economy.

By definition the closed-end fund's cash flows gross of wages equals $n_{it} d_{t+1,1}$ and its value $n_{it} p_{t+1,1}$ in period $t+1$. Thus, the investor's terminal wealth equation (3) becomes

$$w_{t+1}(i) = F'_{t,-1}(i) \left[N_{t+1,-1} \cdot P_{t+1,-1} + N_{t,-1} \cdot D_{t+1,-1} - G_{t+1,-1} \right] + f_{it} \left[n_{t+1,1} p_{t+1,1} + n_{it} d_{t+1,1} - g_{t+1,1} \right] + (1+r) \beta_t(i). \quad (50)$$

By definition the capital base of a closed-end fund equals its holdings in other firms (\hat{F}_t) and its own production capital (m_t) which represents its computer equipment, copy machines and other items used to manage the fund. Thus, by definition the following equalities must hold for the fund's valuation

$$n_{it} p_{it} = \hat{F}'_{t,-1} (N_{t,-1} \cdot P_{t,-1}) + m_t p_t^m \quad (51)$$

and its cash flow gross of wages

$$n_{it} d_{it} = \left(\hat{F}'_{t,-1} - \hat{F}'_{t+1,-1} \right) N_{t+1,-1} \cdot P_{t+1,-1} + \hat{F}'_{t,-1} (N_{t,-1} \cdot D_{t+1,-1} - G_{t+1,-1}) + m_t d_{t+1}^m \quad (52)$$

where a “-1” indicates a vector absent its first element, p_t^m the market price of the closed-end fund’s production capital, and d_{t+1}^m the cash flows deriving from that same capital. Based upon the above equations the investor’s terminal wealth. Now use the above two equations to substitute out for $n_{it}p_{it}$ and $n_{it}d_{it}$ in equation (50) and then rearrange to yield

$$w_{t+1}(i) = \left(F_{t,-1}(i) + f_{it} \hat{F}_{t+1,-1} \right)' \left[N_{t+1,-1} \cdot P_{t+1,-1} + N_{t,-1} \cdot D_{t+1,-1} - G_{t+1,-1} \right] + f_{it} \left[m_{t+1,1} p_{t+1,1}^m + m_{it} d_{t+1,1}^m - g_{t+1,1} \right] + (1+r) \beta_t(i). \quad (53)$$

If one now redefines F_t , N_t , and P_t so that they equal $(f_{it}, F_{t,-1}(i) + f_{it} \hat{F}_{t,-1})'$, $(m_t, N_{t,-1})'$, $(p_t^m, P_{t,-1})'$ respectively then (53) reverts back to the original wealth equation (3). However, as suggested in footnote 15, the pricing equations only depend upon the closed-end fund’s production capital m and its associated volatility and not on the underlying holdings.