
While the original article outlines the steps needed to derive the model’s closed form solution, the algebra involved is not trivial. This note is designed to guide interested readers through the steps in much greater detail than is appropriate given the space constraints for a typical journal article. For the most part, this guide does not include the motivation behind the model’s various assumptions. Interested readers should consult the published paper for it.

1. Define the Laws of Motion and the Objective Functions

There are $n$ firms in a heterogeneous goods industry competing with each other for market share within an infinite horizon setting. Designate firm $i$’s market share at time $t$ as $m_i(t)$. These market shares add to 1 at every instant in time, $\sum_i m_i(t) = 1$. Firms can increase (or defend) their market shares by expending resources on customer acquisition $u_i(t)$. These expenditures may include items like advertising, product development and opening new outlets. Based on what a firm and its competitors spend the law of motion governing the change in market shares is

$$ dm_i = \phi \left[ \frac{(1-m_i)u_is_j - m_i \sum_{j \neq i} u_j s_j}{\sum_{j=1}^n u_j s_j} \right] dt. \tag{1} $$

The variable $\phi$ determines how readily consumers will switch from one brand to another. High values imply consumers are easily persuaded to switch brands, low values imply that it will be quite difficult. If $\phi$ equals zero then consumers never switch and the $dm_i$ are all zero no matter what each firm spends on
customer acquisition. The \( s_i \) terms indicate how cost effective each firm is at acquiring new customers. A firm with a relatively high value can spend less than its rivals with the same overall effect. The terms in the numerator imply that the smaller a firm’s market share the easier it will be for it to acquire new customers. When a firm is small it is initially peeling away customers that have a strong affinity for its own product. As it grows, additional market share can only be obtained by taking customers that might normally be attracted to a rival’s product which makes spending to acquire them more and more expensive. The denominator eliminates units and implies that spending to acquire customers depends on what each firm spends relative to its rivals. Thus, if all firms double their expenditures the \( dm_i \) terms would all remain unchanged. This also has the advantage of guaranteeing that the law of motion in (1) is unit free. Calculating the \( u_i \) in dollars yields the same answer as calculating it in yen.

It will occasionally be useful to rewrite (1) in the following form:

\[
dm_i = \phi \left[ \frac{u_i s_i}{\sum_{j=1}^{n} u_j s_j} - m_i \right] dt. \tag{2}
\]

The intermediate step is to write the numerator in (1) as \( \phi \left[ u_i s_i - m_i \sum_{j=1}^{n} u_j s_j \right] \). Notice that moving the \( u_i s_i \) term into the numerator’s summation then implies it goes from 1 to \( n \) and no longer from 1 to \( n \) exclusive of \( i \).

Corporate profits \( \pi_i(t) \) increase linearly in market share at a rate \( \alpha_i \). The firm also incurs an instantaneous fixed cost of operations \( f_i \) that must be paid no matter what its market share may be. The market, overall, grows at a rate \( g \) and it is assumed that this impacts the overall cost of doing business. Thus, profits at some date \( t \) equal

\[
\pi_i(t) = e^{\alpha_i} \left( \alpha_i m_i(t) - u_i(t) - f_i \right). \tag{3}
\]
Firms are risk neutral and seek to maximize the present discounted value of their profit stream (equation (3)) under the discount rate $r$

$$V_i = \int_{t=0}^{\infty} e^{(g-r)t} \left( \alpha_i m_i(t) - u_i(t) - f_i \right) dt$$

(4)

To simplify the subsequent analysis define $\delta$ as $r - g$. Due to the fixed costs in (3) each firm does benefit from scale economies. What stops the industry from turning into a monopoly is that as a firm’s market share increases the cost of increasing it yet further (by spending resources on customer acquisition $u_i$) goes up as well.

2. **Set up the Hamilton-Jacobi-Bellman Equation**

Setting up and solving the Hamilton-Jacobi-Bellman equation involves two steps. The first is solving a maximization problem that sets the value of each player’s control variable. In this case the $u_i$.

Thus, the goal of this step is to obtain a characterization of the $u_i$ in terms of the model’s exogenous variables. The second step takes the solution for the $u_i$ and inserts them into a differential equation that has to be solved for the firm’s value function. This value function characterizes the firm’s market value at each point in time given the values of the current state variables, which in this model are the $m_i$.

Since the model contains $n$ state variables (the $m_i$) the value function $V_i$ for each firm $i$ may depend upon every company’s market share save one (since the market shares have to add to 1). In this case the Hamilton-Jacobi-Bellman (HJB) equation looks like:

$$0 = \max_{u_i} \alpha_i m_i - u_i - f_i + \sum_{j=1}^{n} \frac{\partial V_i}{\partial m_j} \left[ dm_j \right] - \partial V_i.$$

(5)

Typically problems with multiple state variables are not amenable to closed form solutions. However, as with most problems involving HJB equations, solutions are found by making clever guesses about the
value function’s overall form. Here the way forward is to guess that a firm’s value function only depends upon its own market share. If true, the resulting set of differential equations and their solutions will satisfy (5) for all \( n \) firms simultaneously.\(^1\) Assuming a firm’s value function just depends on its own market share reduces (5) to

\[
0 = \max_{u_i} \alpha m_i - u_i - f_i + \frac{\partial V_i}{\partial m_i} \left( \phi \left( \frac{u_i s_i}{\sum_{j=1}^{n} u_j s_j} - m_i \right) \right) - \partial V_i.
\]

(6)

3. Obtain and Simplify the First Order Conditions

The initial step towards solving the model is to solve for the \( u_i \) in terms of the exogenous variables. This can be accomplished by first solving for each firm’s optimal policy \( u_i \) in terms of its value function \( V_i \).

Differentiate (6) with respect to the \( u_i \), set the result to 0 and then use some minor algebra to produce:

\[
V_i' \phi s_i \left( \sum_{j=1}^{n} s_j u_j \right) = \left( \sum_{j=1}^{n} s_j u_j \right)^2.
\]

(7)

To make the notation somewhat more compact, \( V_i' \) has been used to replace \( \partial V_i / \partial m_i \) in (6). Finding a solution for firm \( i \)’s optimal policy for \( u_i \) directly from (7) is problematic because the equation contains the control variables \( (u_j) \) for all of the firms other than \( i \) as well. This interlinks all of the solutions. If a closed form solution is possible it will be necessary to somehow decouple all of the equations. Notice, that on both sides of (7) the control variables for the firms \( j \neq i \) are fully contained within the expression \( \sum_{j=1}^{n} s_j u_j \). If one can find a solution for this sum in terms that does not involve any of the \( j \neq i \) control variables the first order equations can potentially be decoupled.

\(^1\) If not, it will be impossible to solve (5) for all \( n \) firms simultaneously.
Notice that the right hand side of (7) is identical for all $n$ firms. Taking firm $i$ as the base firm, equating its left hand side value in (7) with the left hand value of (7) for each of the other firms in the industry produces a set of $n-1$ equations of the form:

$$V'_i s_i \left( \sum_{j \neq i} s_j u_j \right) = V'_i s_i s_i u_i + V'_i s_i \left( \sum_{j \neq i} s_j u_j \right)$$

$$\vdots$$

$$V'_{i-1} s_{i-1} \left( \sum_{j \neq i} s_j u_j \right) = V'_{i-1} s_{i-1} s_i u_i + V'_{i-1} s_{i-1} \left( \sum_{j \neq i} s_j u_j \right)$$

$$\vdots$$

$$V'_{n-1} s_{n-1} \left( \sum_{j \neq i} s_j u_j \right) = V'_{n-1} s_{n-1} s_i u_i + V'_{n-1} s_{n-1} \left( \sum_{j \neq i} s_j u_j \right)$$

(8)

The important point to note in (8) is that you create one equation for each firm other than firm $i$. For example, if $i$ equals 1 then the system above would begin with firm 2 on the right hand side and go all the way to $n$. Similarly, if $i$ equals $n$ the system would begin with firm 1 and run up to $n-1$. The system shown above is what it looks like for $i$ between 2 and $n-1$.

While the system of equations in (8) may look rather daunting, they do not have the squared terms in (7) making their manipulation much easier. Now, sum the equations in (8) to get

$$V'_i s_i \left( \sum_{j \neq i} s_j u_j \right) \frac{1}{V'_j s_j} = (n-1)s_i u_i + (n-2) \sum_{j \neq i} s_j u_j.$$  

(9)
After multiplying each side of the equations in (8) by $\frac{1}{V_j s_j}$, the first term on the right hand side of (9) is simply the sum of the remaining $s_j u_i$ in each of the n-1 equations. The final term in (9) is less obvious from the summation of the final terms in (8), however one can work through a few examples to easily see that it is true. Intuitively, none of the final terms in (8) include firm $i$, meaning the final summation in (9) excludes it as well. In addition, each firm $j$ is absent once from the summation. Since there are a total of $n-1$ summations, that leaves $n-2$ summations in the form $\sum_{j \neq i} s_j u_j$. Recall the current goal is to find an expression for the $\sum_{j \neq i} s_j u_j$ terms that do not include the $u_i$. To do this rearrange (9) to isolate the sums on the left hand side of the equation:

$$\begin{align*}
\sum_{j \neq i} s_j u_j \left[ V'_j s_j \sum_{j \neq i} \frac{1}{V'_j s_j} - (n-2) \right] = (n-1)s_i u_i,
\end{align*}$$

which then becomes

$$\begin{align*}
\sum_{j \neq i} s_j u_j = \left[ V'_j s_j \sum_{j \neq i} \frac{1}{V'_j s_j} - (n-2) \right]^{-1} (n-1)s_i u_i.
\end{align*}$$

This expression can now be used to substitute out the $\sum_{j \neq i} s_j u_j$ terms in (7) in terms that no longer include the $u_i$.

Using (11) to replace the $\sum_{j \neq i} s_j u_j$ terms in (7) produces a formula in which the only remaining policy variable is $u_i$,

$$\begin{align*}
V'_i s_i \left[ V'_i s_i \sum_{j \neq i} \frac{1}{V'_j s_j} - (n-2) \right]^{-1} (n-1)s_i u_i = \left\{V'_i s_i \left[ V'_i s_i \sum_{j \neq i} \frac{1}{V'_j s_j} - (n-2) \right]^{-1} (n-1)s_i u_i\right\}^2.
\end{align*}$$
On the left hand side of (12) there is a $s_iu_i$ term and on the right hand side its square (from inside the curly brackets). Pulling the $s_iu_i$ terms from both sides leaves an equation that is linear in the firm’s policy variable.

Swapping the left and right sides of (12) (to get the surviving $u_i$ variable on the left hand side) and then engaging in some minor algebra results in the following solution for firm $i$’s control variable:

$$u_i \left\{ 1 + \left[V'_i s_i \sum_{j \neq i} \frac{1}{V'_j s_j} - (n - 2) \right] (n - 1) \right\}^2 = V'_i \phi \left[V'_i s_i \sum_{j \neq i} \frac{1}{V'_j s_j} - (n - 2) \right]^{-1} (n - 1).$$

In principle equation (13) provides a solution to the $u_i$ in terms of the model’s exogenous variables since the value functions only depend on them. However, while this true it makes analyzing the model difficult since the value functions still have some unknown characterization. Thus, the next step is to characterize them.

4. Find a Closed Form Solution for the Policy Variables

Equation (13) comes pretty close to providing a closed form solution for the policy variables. Unlike the initial first order condition in (7) each firm’s control variable $u_i$ is isolated within its own equation. But, this still leaves the unknown value functions in the characterization. To solve for the value functions another guess regarding their form is needed. In this case, the proper guess is that the $V_i$ are affine in a firm’s own market share implying $V_i = a_i + b_i m_i$, where the $a_i$ and $b_i$ terms are time independent constants. As noted earlier, if all of the guesses regarding the form of the value functions are true then it will be possible to satisfy all $n$ HJB equations simultaneously. If any of the guesses are false then it will not be possible to satisfy the HJB equations simultaneously.

Assuming that $V_i = a_i + b_i m_i$ one has that $V'_i = b_i$. Since the $V_i$ only appear in (13) as derivatives with respect to the firm’s own market share this means the equation can be rewritten as
While the \( b_i \) are guessed to be constants, they are still endogenous and thus prevent using (14) to
directly generate a solution for the \( u_i \) in terms solely of exogenous variables.

In order to derive the \( b_i \) in terms of exogenous variables one has to go back to the original HJB
equation (6) at this point. Using the assumed functional form for \( V_i \) it can be rewritten as

\[
0 = \alpha_i m_i - u_i^* - f_i + b_i \left( \phi \left( \frac{u_i^* s_i}{\sum_{j=1}^n u_j^* s_j} - m_i \right) \right) - \delta (a_i + b m_i)
\]

(15)

where \( u_i^* \) represents \( u_i \) under the optimal policy rule. Notice that all of the terms in (14) are time
independent. Thus, if it holds it must mean the \( u_i \) are also time independent and thus also independent
of \( m_i \). This implies (15) can only hold if the terms multiplying the \( m_i \) are equal zero for all \( m_i \) and if the
terms independent of \( m_i \) also equal zero. Collecting the terms multiplying the \( m_i \) yields

\[
0 = \alpha_i - b_i \phi - \delta b_i
\]

(16)

which produces a solution for \( b_i \)

\[
b_i = \frac{\alpha_i}{\phi + \delta}.
\]

(17)

This leaves each of the \( b_i \) only in terms of exogenous variables. Thus (17) is its solution given the various
guesses that have been made regarding the form of the value functions.

With the \( b_i \) expressed in terms only involving the exogenous variables it becomes possible to
find the solution for the \( u_i \) and then the \( a_i \) terms. Plug (17) into (14) to eliminate the \( b_i \),
\[
u_i \left\{ 1 + \left[ \frac{\alpha_i s_i}{\phi + \delta} \sum_{j \neq i} \frac{\phi + \delta}{\alpha_j s_j} - (n - 2) \right]^{-1} \right\}^2 \left[ \frac{\alpha_i s_i}{\phi + \delta} \sum_{j \neq i} \frac{\phi + \delta}{\alpha_j s_j} - (n - 2) \right]^{-1} (n - 1) \right\},
\]

(18)

and then simplify it to

\[
u_i \left\{ 1 + \left[ \frac{\alpha_i s_i}{\phi + \delta} \sum_{j \neq i} \frac{1}{\alpha_j s_j} - (n - 2) \right]^{-1} \right\}^2 \left[ \frac{\alpha_i s_i}{\phi + \delta} \sum_{j \neq i} \frac{1}{\alpha_j s_j} - (n - 2) \right]^{-1}. \]

(19)

Equation (19) yields a solution to all of the policy variables solely in terms of exogenous variables and thus provides their closed form solution. To find the solution to the value functions, use (19) to eliminate the \( u_i^* \) terms from the terms in (15) that are independent of the \( m_i \).

While (19) offers a closed form solution to the \( u_i \), it is cumbersome to work with and it is difficult to see what it implies about the industry. To help with these problems it is useful to define a new variable \( z \) as

\[
z = \sum_{j=1}^{n} \frac{1}{\alpha_j s_j}.
\]

(20)

One can think of this as the level of competition in the industry. Firms that gain significant profits from each unit of market share (high value of \( \alpha_i \)) and are relatively efficient at pulling in customers (high value of \( s_i \)) will be particularly fearsome rivals. Equation (20) provides a statistic showing just how tough the overall industry is.

The \( \alpha_i s_i \sum_{j \neq i} 1/\alpha_j s_j \) terms in equation (19) can be rewritten as \( \left( \alpha_i s_i \sum_{j=1}^{n} 1/\alpha_j s_j \right) - 1 \) by adding and subtracting \( \alpha_i s_i /\alpha_i s_i \) from them. Using this relationship (19) now reduces to

\[
u_i \left\{ 1 + \left[ \alpha_i s_i z - (n - 1) \right]^{-1} \right\}^2 \left( \frac{\alpha_i \phi(n - 1)}{(\phi + \delta)\left( \alpha_i s_i z - (n - 1) \right)} \right).
\]

(21)
To express the \( u_i \) in a much more tractable form, expand the fractions on the left hand side of (21) to generate

\[
\begin{aligned}
\nu_i \left\{ \frac{1 - n + \alpha_i s_i z + n - 1}{1 - n + \alpha_i s_i z} \right\}^2 &= \frac{\alpha_i \phi(n - 1)}{(\phi + \delta)(1 - n + \alpha_i s_i z)}.
\end{aligned}
\] (22)

One last step yields the expression used in most of the paper:

\[
\begin{aligned}
u_i &= \frac{\alpha_i \phi(n - 1)(1 - n + \alpha_i s_i z)}{(\alpha_i s_i z)^2 (\phi + \delta)}.
\end{aligned}
\] (23)

To solve for the constants \( a_i \) in the value function (15) equate the terms that are independent of \( m_i \) to zero and substitute out the \( u_i^* \) for the solution in (23). This implies

\[
\begin{aligned}
0 &= -\frac{\alpha_i \phi(n - 1)(1 - n + \alpha_i s_i z)}{(\alpha_i s_i z)^2 (\phi + \delta)} - \sum_{j=1}^{\infty} \frac{\alpha_j \phi(n - 1)(1 - n + \alpha_j s_j z)}{(\alpha_j s_j z)^2 (\phi + \delta)} - \delta a_i.
\end{aligned}
\] (24)

Some minor algebra then reduces (24) to

\[
\begin{aligned}
\delta a_i &= -\frac{\alpha_i \phi(n - 1)(1 - n + \alpha_i s_i z)}{(\alpha_i s_i z)^2 (\phi + \delta)} - \sum_{j=1}^{\infty} \frac{\alpha_j \phi(n - 1)(1 - n + \alpha_j s_j z)}{(\alpha_j s_j z)^2 (\phi + \delta)} - \frac{\phi a_i}{\phi + \delta} (1 - n + \alpha_i s_i z).
\end{aligned}
\] (25)

Combining the terms that do not involve \( f_i \) then produces the solution to the \( a_i \) used in the paper,

\[
\begin{aligned}
a_i &= \frac{\alpha_i \phi(1 - n + \alpha_i s_i z)^2}{\delta (\alpha_i s_i z)^2 (\phi + \delta)} - \frac{f_i}{\delta}.
\end{aligned}
\] (26)
Thus, equation (23) characterizes each firm’s spending on customer acquisition, equation (26) the constant in the firm’s value function and equation (17) the value obtained by a firm from each fractional point of market share.