On Equilibrium Asset Price Processes

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In this article we derive necessary and sufficient conditions that must be satisfied by equilibrium asset price processes in a pure exchange economy. We examine a world in which asset prices follow a diffusion process, asset markets are dynamically complete, all investors maximize their (state-independent) expected utility of consumption at some future date, and investors have nonrandom exogenous income. We show that it is necessary and sufficient that the coefficients of an equilibrium diffusion price process satisfy a partial differential equation and a boundary condition. We also examine how the dynamics of asset prices are related to the shape of the representative investor's utility function through the boundary condition. For example, in a constant-volatility economy, the expected instantaneous return of the market portfolio is mean reverting if and only if the relative risk aversion of the representative investor is decreasing in terminal wealth.

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This article is a revised version of an earlier one, entitled "Equilibrium Asset Price Processes". We thank seminar participants at HEC, UC Berkeley, UCLA, University of Minnesota, and the University of Pennsylvania-Wharton for helpful comments. We are also grateful to the referee (S. Maheswaran) and editor Phil Dybvig for many helpful comments and suggestions, and thank executive editor Chester Spatt for editorial suggestions. Financial support from the Batterymarch Fellowship Program (for Hua He) and the Berkeley Program in Finance is gratefully acknowledged. Address proofs and reprint requests to Hua He, Haas School of Business, University of California at Berkeley, Berkeley, CA 94720.

The Review of Financial Studies 1993 Volume 6, number 3, pp. 593–617

examine a world in which asset prices follow a diffusion process, asset markets are dynamically complete, all investors maximize their (state-independent) expected utility of consumption at some future date, and investors have nonrandom exogenous income. The assumption of dynamically complete markets allows us to work in a representative investor framework [see Constantinides (1982)].

We show that for a diffusion process to be an equilibrium asset price process of the market, it is necessary and sufficient that the coefficients of the diffusion process satisfy a partial differential equation and a boundary condition. The partial differential equation is derived from the fact that the representative investor optimally holds the market portfolio and therefore follows a path-independent strategy. We also examine how the dynamics of asset prices are related to the shape of the representative investor's utility function through the boundary condition. For example, in a constant-volatility economy, the expected instantaneous return of the market portfolio is mean reverting if and only if the relative-risk aversion of the representative investor is decreasing in terminal wealth.

Our assumptions of diffusion processes for asset prices and state-independent utility functions for investors are common in the finance literature: see Merton (1971, 1973) and Breeden (1979).[1] Economies with dynamically complete markets have also been widely studied, as illustrated in Black and Scholes (1973). The assumption of nonrandom exogenous income is special but later will be weakened.

The results of this article supplement those of the international capital asset pricing models (ICAPM) developed by Merton (1973) and extended by Breeden (1979). These models assume that asset prices and a vector of exogenously specified state variables follow a multidimensional diffusion process. With this assumption, equilibrium conditions are then imposed that restrict the expected instantaneous returns of individual assets relative to the expected instantaneous return of the market portfolio. Since the ICAPM emphasizes only the relative pricing between individual assets and the market portfolio, these models have not provided a full characterization of the asset price processes consistent with a market equilibrium.[2] With our (additional) assumptions of the economy, we are able to provide a complete characterization of equilibrium price processes.[3]

[1] Huang (1987) provides a theoretical foundation for the system of equilibrium prices and state variables to form a diffusion process in a pure exchange economy. The sufficient conditions involve certain properties of investors' utility functions, the aggregate endowment process, and the dividend processes of the traded assets.

[2] Rubinstein (1976), Breeden and Litzenberger (1979), and Brennan (1979) have shown that a lognormal process for the market portfolio is consistent with a representative investor having constant relative-risk aversion. Gennotte and Marsh (1993) extend the analysis to allow random volatility of the dividend process; the price process no longer remains lognormal.

Our article is closely related to Bick (1990), who also presents a set of necessary and sufficient conditions for a diffusion price process to be supported by an economy similar to ours. But, our approach is quite different from, and simpler than that of, Bick. In order to verify that a given process is an equilibrium process, Bick's approach involves computing the conditional expectations of the marginal utility of consuming the terminal value of assets at the final date, which could be very difficult to calculate if the conditional density function does not have an analytical form. However, in the special case where asset prices are time-homogeneous diffusions, Bick simplifies his conditions so that computing conditional expectations is no longer required. Our necessary and sufficient conditions in this special case are the same as Bick's.

Our article is also related to Wang (1991), who studies equilibrium conditions for asset price processes in an economy similar to ours, except that Wang allows intermediate consumption. Wang's approach relies upon the existence of a solution to a certain differential equation, an approach that is also different from ours. While Wang derives a set of sufficient conditions for a given price process to be an equilibrium process, these conditions are much stronger than necessary. Therefore, a complete characterization of equilibrium price process is not provided. However, we can apply our approach to Wang's economy to get conditions that are both necessary and sufficient.

The rest of the article is organized as follows. Section 1 formulates a continuous-time securities market–pure exchange economy in a single representative investor setting. Section 2 derives the necessary and sufficient conditions for a diffusion price process of the market to be an equilibrium in our economy. Examples are also provided to illustrate these conditions. In Section 3 we examine an economy in which the volatility of asset returns is a constant. Section 4 contains some concluding remarks.

1. Formulation

Consider a continuous-time dynamically complete securities market–pure exchange economy in which there is a single representative investor who has a finite lifetime horizon $[0, T]$. There is one risky stock and one riskless bond available for trading at any time between 0 and T. The risky stock can be viewed as the market portfolio, the total supply of which is normalized to one share. The riskless bond

[1] Had we allowed arbitrary state-dependent utility functions or arbitrary exogenous income processes, then arbitrary arbitrage-free price systems could be supported by competitive equilibrium [see Kreps (1981)].

is viewed as a financial asset, which is in zero net supply.[4] We assume that the price process for the stock S is a diffusion process and can be described by the stochastic differential equation

$$\frac{dS(t)}{S(t)} = \mu(S(t), t)\, dt + \sigma(S(t), t)\, dw(t), \qquad t \in [0, T], \tag{1}$$

where μ and σ are twice continuously differentiable with respect to S and continuously differentiable with respect to t, and w is a standard Brownian motion defined on a complete probability space $(\Omega, P, \mathcal{F})$.[5] It is assumed that $\sigma > 0$ almost surely and that the stock price is strictly positive with probability 1. Thus, each $\omega \in \Omega$ specifies a complete history of the Brownian motion as well as the stock price. We assume that the stock pays no dividends. To avoid the potential difficulty of classifying boundary behavior, we further assume that S can take values on the entire positive real line and S cannot be negative. For simplicity, the equilibrium interest rate for the bond is taken to a constant, r.[6] The representative investor is assumed to have access only to the information contained in the historical prices, which can be modeled by the σ-field generated by $\mathcal{F}_t = \sigma\{S(s); 0 \leq s \leq t\}$ for $t \in [0, T]$ with $\mathcal{F} = \mathcal{F}_T$.

We assume that there exists an equivalent martingale measure or a risk-neutral probability Q for the stock-price process considered in (1).[7] This equivalent martingale measure is defined as

$$Q(A) \equiv \int_A \xi(\omega, T) P(d\omega) \qquad \forall\, A \in \mathcal{F}, \tag{2}$$

[4] See Remark 1(C) for the case with a positive net supply of the riskless bond.

[5] Implicit in this is the assumption that a solution to the stochastic differential equation (1) exists. The assumption that the equilibrium price process for the stock is a diffusion process can be derived from a set of more primitive assumptions. For example, one can view the stock as a claim to an endowment or crop to be received at the final date T, where the endowment is taken to be a numeraire good. The size of the endowment or crop follows a diffusion process

$$dz(t) = \alpha(z(t), t)\, dt + \beta(z(t), t)\, dw(t).$$

If we assume that $z(t)$ is observable to the investor at time t and that the investor has a von Neumann–Morgenstern, state-independent utility function, then the equilibrium stock-price process must be a function of z and t and, consequently, should have the form as assumed here. See Huang (1987) for more details. Wang (1991) works directly with the condition that the equilibrium price should be a function of z and provides sufficient conditions for this function to become an equilibrium price.

[6] Since there is no intermediate consumption in our model, the riskless interest rate cannot be determined in equilibrium and is therefore specified exogenously.

[7] See Harrison and Kreps (1979) for the former and Cox and Ross (1976) for the latter.

where

$$\xi(\omega, t) = \exp\left\{\int_0^t -\frac{\mu(S(s), s) - r}{\sigma(S(s), s)}\, dw(s) - \frac{1}{2}\int_0^t \left(\frac{\mu(S(s), s) - r}{\sigma(S(s), s)}\right)^2 ds\right\}. \tag{3}$$

The $\xi(\omega, t)e^{-rt}$ can also be interpreted as the Arrow–Debreu state price (at time 0) per unit of probability for one unit consumption good to be received at state ω and time t. We will sometimes call $\{\xi(t)\}$ state prices without mentioning interest rate discounting and per unit of probabilty. Under the equivalent martingale measure Q, the stock-price dynamics becomes

$$\frac{dS(t)}{S(t)} = r\, dt + \sigma(S(t), t)\, dw^*(t),$$

where $w^*(t) \equiv w(t) + \int_0^t (\mu(s) - r)/\sigma(s)\, ds$ is a standard Brownian motion under Q. Given our current setup, the equivalent martingale measure must be unique [see Harrison and Kreps (1979)].

We consider a representative investor who consumes only at the final date[8] and whose preferences for consumption at the final date can be represented by the expected utility of a von Neumann–Morgenstern, state-independent utility function $\mathbf{E}U(W(T))$, where $W(T)$ denotes the wealth at the final date, and U is twice continuously differentiable, increasing, and concave. For now, we assume that the representative investor does not receive any exogenous income,[9] is endowed with one unit of the stock at time 0, and is allowed to allocate the wealth between the stock and the bond so that he maximizes the expected utility of consuming the final wealth at the final date. That is, he solves the dynamic consumption and investment problem

$$\begin{aligned} &\sup_A \mathbf{E}U(W(T)) \\ \text{s.t.}\quad & dW(t) = (rW(t) + A(t)(\mu(S(t), t) - r))\, dt \\ &\qquad\qquad + A(t)\sigma(S(t), t)\, dw(t), \qquad t \in [0, T], \\ & W(t) \geq 0, \qquad t \in [0, T], \end{aligned} \tag{4}$$

where $A(t)$ denotes the dollar amount invested in the stock at time t. The first constraint in (4) is the dynamic budget constraint determining the evolution of the wealth process. The second constraint in (4) is the nonnegative wealth constraint, which rules out the pos-

[8] We will discuss the effect of intermediate consumption in Section 2.

[9] We will relax this assumption in Section 2.

sibility of creating something out of nothing [see Dybvig and Huang (1988)]. A dynamic investment strategy is said to be an equilibrium investment strategy if it requires that the investor optimally invest all the wealth in the risky stock at each moment in time and consume the terminal value of the stock at the final date. In this case, the stock-price process considered in (1) is said to be an equilibrium in our economy.

Our definition of equilibrium is stronger than what we usually mean by competitive equilibrium. Specifically, we have assumed that U is state-independent, the representative investor does not receive exogenous income, and consumption occurs at the final date T. Thus, the class of asset price processes considered here is only a subclass of equilibrium asset price processes consistent with competitive equilibrium.

2. Characterization of Equilibrium Price Processes

In this section we characterize equilibrium asset price processes for the economy specified in the previous section. The main idea of our analysis is to exploit the condition that in equilibrium the representative investor follows a path-independent strategy (i.e., holding the market portfolio). We derive necessary and sufficient conditions for a given price process to be an equilibrium in our economy. We also provide examples that show how our results can be used to identify equilibrium asset prices.

Let A be the investment function that solves the investor's dynamic consumption and investment problem, and let $J(W, S, t)$ be the value of the optimal objective function or the *indirect utility function,* given that the wealth and the stock price at time t are W and S, respectively. Assume that J is twice continuously differentiable with respect to W and S and continuously differentiable with respect to t for $W > 0, S > 0$, and $t \in (0, T)$. Then, following Merton (1971, 1973), J must satisfy the Bellman equation

$$0 = \max_A \Big\{ J_t + (rW + A(\mu - r))J_W + \mu S J_S \\ \tfrac{1}{2}\sigma^2 A^2 J_{WW} + \sigma^2 SAJ_{WS} + \tfrac{1}{2}\sigma^2 S^2 J_{SS} \Big\}, \tag{5}$$

for all $W > 0, S > 0$, and $t \in (0, T)$, and the boundary conditions

$$\lim_{t \uparrow T} J(W, S, t) = U(W) \qquad \text{and} \qquad \lim_{W \downarrow 0} J(W, S, t) = U(0),$$

where all subscripts denote partial derivatives. Note that the second boundary condition reflects the nonnegativity constraints on wealth. The first-order condition implies that

$$A(W, S, t) = -\frac{\mu(S, t) - r}{\sigma(S, t)^2}\frac{J_W(W, S, t)}{J_{WW}(W, S, t)} - S\frac{J_{WS}(W, S, t)}{J_{WW}(W, S, t)}, \quad (6)$$

where the first term on the right-hand-side is an instantaneous mean-variance efficient portfolio, and the second term represents the hedging demands against adverse changes in the consumption–investment opportunity set. In equilibrium, we have

$$A(S(t), S(t), t) = S(t),$$

because there is only one unit of the stock available for trading. We should restrict our attention only to those price processes and utility functions so that J are continuously differentiable with respect to W and S up to the fourth order and that J, J_W, J_{WW}, and J_{WS} are continuously differentiable with respect to t. This allows us to work with many derivatives of the indirect utility function. Differentiating (5) with respect to W gives

$$0 = J_{Wt} + rJ_W + (rW + A(\mu - r))J_{WW} + \mu S J_{WS}$$
$$+ \tfrac{1}{2}\sigma^2 A^2 J_{WWW} + A\sigma^2 S J_{WWS} + \tfrac{1}{2}\sigma^2 S^2 J_{WSS},$$

where we have used (6) to simplify terms. This equation implies that the drift of dJ_W is $-rJ_W$. The diffusion term of dJ_W is $J_{WW}\sigma A + J_{WS}\sigma S$, which equals $-((\mu - r)/\sigma)J_W$ by (6). We conclude that

$$dJ_W(W(t), S(t), t)$$
$$= -rJ_W\,dt - \frac{\mu(S(t), t) - r}{\sigma(S(t), t)}J_W(W(t), S(t), t)\,dw(t).$$

Solving this stochastic differential equation gives

$$J_W(W(t), S(t), t)$$
$$= J_W(0)e^{-rt}\exp\left\{\int_0^t -\frac{\mu(S(s), s) - r}{\sigma(S(s), s)}\,dw(s) - \frac{1}{2}\int_0^t \left(\frac{\mu(S(s), s) - r}{\sigma(S(s), s)}\right)^2 ds\right\}$$
$$= J_W(0)\xi(t)e^{-rt},$$

where $J_W(0) = J_W(W(0), S(0), 0)$ and $\xi(t)$ is defined in (3). Since in equilibrium the representative investor holds one share of the risky stock, we have $W(t) = S(t)$ and

$$J_W(S(t),S(t),t) = J_W(S(0),S(0),0)\xi(t)e^{-rt}.$$

The previous equilibrium condition implies that the Arrow–Debreu state price at time t, $\xi(t)$, must be path independent (i.e., independent of the past history of stock prices) for all t. Since the state-price process at time t, $\xi(t)$, depends in general on the historical stock prices through the integral of $(\mu - r)/\sigma$ with respect to dw, path independence clearly puts stringent conditions on μ and σ. The equilibrium condition also implies that $U'(S(T)) = J_W(0)\xi(T)e^{-rT}$, which requires in equilibrium that holding the total supply of the risky stock be optimal and that the state price at time T and the stock price be inversely related.

The following theorem explores the foregoing discussion and provides necessary and sufficient conditions for the stock-price process to satisfy in equilibrium in our economy.

Theorem 1. *The necessary and sufficient conditions for $\{S_t, t \in [0, T]\}$ defined in (1) to be in equilibrium in our economy with a single representative investor consuming at a fixed final date are as follows:*

(i) (μ, σ) satisfies the partial differential equation

$$\tfrac{1}{2}\sigma^2 S^2 f_{SS} + \mu S f_S + f_t + \sigma\sigma_S S(S f_s + f^2 - f) = 0, \qquad (7)$$

where $f(S, t) = (\mu(S, t) - r)/\sigma(S, t)^2$.

(ii) There exists an increasing and concave utility function U such that f satisfies the boundary condition

$$f(S, T) = -SU''(S)/U'(S). \qquad (8)$$

Remark 1. (A) If we let $\mathcal{L}$ be the differential generator associated with S—that is, $\mathcal{L}(f) \equiv \frac{1}{2}\sigma^2 S^2 f_{SS} + \mu S f_S$—then (7) can be rewritten as

$$\mathcal{L}f + f_t + \sigma\sigma_S S(Sf_S + f^2 - f) = 0.$$

(B) Condition (ii) is essentially equivalent to the condition that $\mu(S, T) \geq r$. To verify the existence of U that satisfies (8), it is sufficient to require that $f(S, T) \geq 0$ for all $S > 0$ and that $f(x, T)/x$ is integrable on any closed interval in $(0, \infty)$ or, equivalently, $f(e^x, T)$ is integrable on any closed interval in $(-\infty, +\infty)$. The utility function that satisfies (8) is determined by

$$U'(S) = \gamma \exp\left(-\int_{S_0}^{S} \frac{f(x, T)}{x}\, dx\right), \qquad S > 0,$$

or

$$U'(S) = \gamma \exp\left(-\int_{\ln S_0}^{\ln S} f(e^x, T)\, dx\right), \qquad S > 0,$$

where γ and S_0 are positive constants. Integrating both sides with respect to S yields the utility function U that satisfies (8).

(C) If the total supply of the riskless bond is strictly positive, say $\bar{B} > 0$, then we have

$$J_W(S(t) + \bar{B}e^{rt}, S(t), t) = J_W(S(0) + \bar{B}, S(0), 0)\xi(t)e^{-rt}.$$

Obviously, f shall satisfy the same partial differential equation (7) with the boundary condition

$$f(S, T) = -SU''(S + \bar{B}e^{rT})/U'(S + \bar{B}e^{rT}).$$

Proof. We prove the necessity here, while leaving the proof for the sufficiency to the Appendix. Define $h(S(t), t) = \ln J_W(S(t), S(t), t) - \ln J_W(S(0), S(0), 0)$. Since $J_W(t) = J_W(0)\xi(t)e^{-rt}$ in equilibrium, we have $h(S(t), t) = \ln \xi(t) - rt$. Hence, the drift and diffusion terms of dh must be the same as those of $d(\ln \xi(t) - rt)$. Applying Itô's lemma, we obtain

$$\tfrac{1}{2}\sigma^2 S^2 h_{SS} + \mu S h_S + h_t = -\tfrac{1}{2}((\mu - r)/\sigma)^2 - r,$$

$$\sigma S h_S = -(\mu - r)/\sigma,$$

or, equivalently,

$$S h_S = -f, \tag{9}$$

$$\tfrac{1}{2}\sigma^2 S^2 h_{SS} - \tfrac{1}{2}\sigma^2 f^2 + rf + r + h_t = 0. \tag{10}$$

Now, differentiating (9) with respect to S and t, we get $h_{SS}S + h_S = -f_S$, and $Sh_{St} = -f_t$. Hence, $S^2 h_{SS} = -Sf_S + f$, and (10) becomes

$$\tfrac{1}{2}\sigma^2(-Sf_S + f) - \tfrac{1}{2}\sigma^2 f^2 + rf + r + h_t = 0. \tag{11}$$

Next, differentiating (11) with respect to S and substituting $Sh_{St} = -f_t$ into the resulting equation, we get (7). Finally, the boundary condition for f is obviously satisfied if we set $t = T$ in (6). ■

The partial differential equation (7) is an equilibrium condition imposed on the coefficients of the stock-price process. The argument that is crucial to the derivation of this equation is the observation that h is a function of $S(t)$ and t but not of the history of S. Thus, (7) is equivalent to that the state-price process at time t, $\xi(t)$, is path independent for all t. Consequently, (7) guarantees that for any investor with a state-independent utility function, he will follow a path-independent investment strategy. Cox and Leland (1982) have shown that any optimal investment strategy must be path independent if the stock-price process follows a lognormal process. Our result extends that of Cox and Leland to a more general class of diffusion processes. The boundary condition (8) ensures that holding the total supply of the risky stock is optimal for an investor with a utility function U.

Theorem 1 provides a complete characterization of equilibrium asset price processes within the family of diffusion processes. We obtain this characterization by exploiting the equilibrium condition that the representative investor optimally holds the total supply of the risky stock. Note that this characterization cannot be derived directly from the ICAPM. Consequently, our equilibrium conditions are stronger than those derived in the ICAPM.

If we are given a pair of diffusion coefficients (μ, σ), then we can easily check whether it satisfies the necessary conditions of Theorem 1. If any of these necessary conditions is violated, then we can immediately conclude that S can never be an equilibrium in our economy.

Theorem 1 also suggests that for a given volatility function σ and a given utility function U the expected instantaneous excess-return function normalized by the volatility function, f, is determined by a nonlinear partial differential equation subject to a boundary condition at $t = T$. Under some standard regularity conditions, the boundary value problem defined by (7) and (8) has a unique solution [see Friedman (1969)]. Thus, for given σ and U, one can derive f by solving the partial differential equation (7), which can then be used to find the expected instantaneous return function μ. Clearly, the expected instantaneous return function depends on the shape of the utility function in an important way through the boundary condition. Of course, in order for a price process to be an equilibrium process, we still have to make sure that S is strictly positive.

Similarly, for a given expected instantaneous return function μ, we can rewrite (7) as

$$\frac{1}{2}S^2 f_{SS} + Sff_S + \frac{rf^2}{\mu - r} + \frac{1}{\mu - r}ff_t + \frac{S}{2}\left(\frac{\mu_S}{\mu - r} - \frac{f_S}{f}\right)(Sf_S + f^2 - f) = 0. \tag{12}$$

Thus, we can alternatively solve (12) for f and thereby find the volatility function σ. In this case, we have to make sure that $\sigma^2 = (\mu - r)/f$ is strictly positive. In summary, if we specify either the volatility function or the expected instantaneous return function, then the expected instantaneous return function or the volatility function can be determined by (7) or (12) and by the corresponding boundary condition. Theorem 1 suggests that for any given utility function, there exists a fairly large class of the expected instantaneous return and volatility functions that are consistent with our definition of equilibrium. Equation (7) or (12) can be used to find systematically all of the equilibrium expected instantaneous return and volatility functions.

Bick (1990) has obtained a characterization of equilibrium or *viable* price processes for the same class of diffusion processes studied in this article. For a given pair of diffusion coefficients (μ, σ), Bick constructs the utility function in the same manner as we did in Remark 1 and shows that the price process S is consistent with an equilibrium if and only if

$$S(t) = E[U'(S(T))S(T)\,|\,\mathcal{F}_t]/E[U'(S(T))\,|\,\mathcal{F}_t],$$

where $\mathcal{F}_t$ denotes the information set generated from the historical stock prices. Therefore, in order to verify whether S is consistent with an equilibrium, we must compute conditional expectations. Since the conditional density functions are difficult to calculate in general, the necessary conditions derived in Propositions 1 and 2 of Bick (1990) can hardly be verified for general diffusion processes. In contrast, the necessary conditions derived in Theorem 1 are easy to verify for any arbitrarily given diffusion process. Bick's (1990) condition (P2) in Proposition 2 states that in equilibrium, for any $0 \leq t_1 < \ldots < t_n \leq T$ and $S_0, S_1, \ldots, S_n > 0$,

$$k(t_0, S_0; t_1, S_1)k(t_1, S_1; t_2, S_2) \ldots k(t_{n-1}, S_{n-1}; t_n, S_n) = k(t_0, S_0; t_n, S_n),$$

where $k(t, x; s, y) = p(t, x; s, y)/q(t, x; s, y)$, and p and q are the transition density functions under probability measures P and Q, respectively. This condition is essentially equivalent to that ξ is path independent as $dQ/\mathrm{d}P = \xi(T)$.

Our approach is also quite different from that of Wang, who works directly with a dividend process and derives equilibrium conditions on the function that maps from the value of current dividend to the asset price. However, Wang's general characterization of the equilibrium price contains an unknown utility function that makes it difficult to verify. By working directly with the dynamics of the equilibrium asset price process, we derive explicit conditions that are easy to verify.

We now focus on an important subclass of diffusion processes that are of particular interest to financial economists, namely the class of time-homogeneous diffusion processes,

$$\frac{dS(t)}{S(t)} = a(S(t))\ dt + b(S(t))\ dw(t), \tag{13}$$

where $a \geq 0$ and b are functions of S. To check whether such a process is consistent with an equilibrium, one can verify whether f satisfies (7). Since in this special case f is independent of t, or $f_t = 0$, (7) is equivalent to

$$\frac{d}{dS}(\sigma^2[Sf_S + f^2 - f]) = 0.$$

That is, there exists a constant K such that $\sigma^2[Sf_S + f^2 - f] = K$. This is the same condition that Bick (1990) obtains in Proposition 3 for time-homogeneous diffusions. We summarize the necessary conditions for this special case in the following corollary.

Corollary 1. The necessary and sufficient conditions for $\{St_t,\ t \in [0, T]\}$ defined in (13) to be consistent with an equilibrium are as follows:

(i) There exists a constant K such that

$$\sigma^2[Sf_S + f^2 - f] = K, \tag{14}$$

where $f(S) = (\mu(S) - r)/\sigma^2(S)$.

(ii) There exists an increasing and concave utility function U such that $f(S) = -SU''(S)/U'(S)$.

To illustrate the use of Theorem 1 and Corollary 1, we consider one example. More examples with time-homogeneous processes can be found in Bick (1990).

Example 1. Assume $r = 0$. Consider the class of diffusion processes defined by

$$\frac{dS(t)}{S(t)} = a(S(t))\ dt + b\ dw(t),$$

where b is a constant and a is a function of S. We have $\mu = a(S)$, $\sigma = b$, and $f = a(S)/b^2$. Thus,

$$\sigma^2(Sf_S + f^2 - f) = Sa_S + a^2/b^2 - a.$$

For the right-hand side of the equation to be a constant, it is necessary that

$$a(S) = \alpha\frac{AS^{2\alpha} - 1}{AS^{2\alpha} + 1} + \frac{1}{2}b^2$$

or

$$a(S) = \alpha \tan(AS^{-\alpha}) + \tfrac{1}{2}b^2$$

for some $\alpha > 0$ and $A > 0$. Since we require $a > 0$, the second class of solutions is not useful. For the first class of solutions, we require $\alpha \leq \frac{1}{2}b^2$.

It is important to point out that when we restrict the price processes to the class of time-homogeneous diffusion processes, the volatility function can be determined from the utility function. We can do this because f is now completely determined by the utility function U; that is, $f = -SU''/U'$ and

$$\sigma^2 = K/(Sf_S + f^2 - f).$$

Since K is a constant, the sign of $Sf_S + f^2 - f$ must be constant for all $S > 0$. This clearly puts further restrictions on the utility function. Specifically, it requires that the sign of $-(d/dS)\,(U''(S)/U'(S)) + (U''(S)/U'(S))^2$ be constant for all $S > 0$, as shown in Bick (1990) as well. For example, if $U(S) = \ln S + S^{1-\alpha}/(1-\alpha)$, then

$$-\frac{d}{dS}\left(\frac{U''(S)}{U'(S)}\right) + \left(\frac{U''(S)}{U'(S)}\right)^2 = (\alpha - 1)\left(\frac{\alpha}{S^{2\alpha+2}} - \frac{\alpha-2}{S^{\alpha+3}}\right).$$

For $\alpha > 2$, the right-hand side of this equation can change signs. Hence, there does not exist a time-homogeneous diffusion process that is consistent with our definition of equilibrium under such a utility function. Later, we demonstrate that there exists a non-time-homogeneous diffusion process for this utility function that is consistent with our definition of equilibrium. Thus, in some cases, if one wants to identify the equilibrium price process for a given utility function, one may have to search among processes that are not time homogeneous. In summary, Theorem 1 has provided a general approach that characterizes the equilibrium asset price processes.

There are a number of ways in which we can extend our result. First, we can allow intermediate consumption in our pure exchange economy (with no production). As it turns out, intermediate consumption does not have any significant effect on the characterization of equilibrium price processes as long as we specify the dividend process as a function of the current stock price and time. Let $D(t) = D(S(t), t)$ be the dividend process, and let S satisfy

$$dS(t) = (\mu(S(t), t)S(t) - D(S(t), t))\,dt + \sigma(S(t), t)S(t)\,dw(t).$$

If S can be supported by utility functions (u, U), where $u(x, t)$ and $U(x)$ are increasing and concave in x, u is the utility function for intermediate consumption and U for final wealth, then in equilibrium

$$u'(D(S(t), t)) = J_W(S(t), S(t), t) = J_W(S(0), S(0), 0)\xi(t)e^{-rt}.$$

Defining h as in the proof of Theorem 1, we have

$$\tfrac{1}{2}\sigma^2S^2h_{SS} + (\mu S - D)h_S + h_t = -\tfrac{1}{2}((\mu - r)/\sigma)^2 - r,$$

$$\sigma S h_S = -(\mu - r)/\sigma.$$

Eliminating h in the same way as before, we can show that (μ, σ) must satisfy the partial differential equation

$$\mathcal{L}f + f_t + (D/S)_S Sf + \sigma\sigma_S S(Sf_S + f^2 - f) = 0,$$

where $\mathcal{L}f = \frac{1}{2}\sigma^2S^2f_S + (\mu S - D)f_S$. The boundary condition is the same as in (8). When there is intermediate consumption, we should also have

$$u'(D(S(t), t)) = J_W(0)\xi(t)e^{-rt}.$$

Apply Itô's lemma, we find that the utility function for intermediate consumption must satisfy

$$-\frac{u''(D(S(t), t), t)}{u'(D(S(t), t), t)} S(t) D_S(S(t), t) = \frac{\mu(S(t), t) - r}{\sigma^2(S(t), t)}.$$

Since u is concave, we require that $\mu(S, t) \geq r$ for all $S > 0$ and $t \in [0, T]$ if the dividend rate is increasing in the level of stock price. The above equation also suggests how one should be able to back out u by integration for given μ, σ, and D.

Second, we point out that the class of diffusion processes we examined here presumes that all relevant information to future returns is fully reflected in the current price.[10] However, we can enlarge the class of equilibrium price processes to include Itô processes that have the form

$$\frac{dS(t)}{S(t)} = \mu(\omega, t)\ dt + \sigma(S(t), t)\ dw(t),$$

where ω is the entire stock-price history rather than just the current stock price. We will show that in this case the equilibrium condition implies that μ must depend only on the current stock price and not on the past history of stock prices. Therefore, even if we allow the expected return to depend on the entire history of the stock price, the equilibrium price process must follow a diffusion as assumed in (7) as long as the volatility of return depends only on the current stock price.

This observation is useful, since empirically it is very difficult to estimate the drift term, and yet it is fairly easy to estimate the diffusion term by using finely sampled observations. If one can empirically determine that the diffusion term depends only on the current price and time, then one can claim that stock price must follow a diffusion process.

To demonstrate our claim, we recall the first-order equilibrium condition

$$U'(S(T)) = \lambda\xi(T)e^{-rT},$$

where ξ is now defined similarly as in (3), except that $\mu(S(t), t)$ is replaced by $\mu(\omega, t)$. Thus,

$$\frac{1}{U'(S(T))} = \frac{e^{rT}}{\lambda}\xi(T)^{-1}.$$

[10] Recent studies have suggested that other economic variables may help predict the future return distribution; see, for example, Harvey (1989).

Now, consider processes S and ξ under the equivalent martingale measure Q. Since $w^*(t) = w(t) + (\mu(t) - r)/\sigma(t)$ is a standard Brownian motion under Q, we have

$$\frac{dS(t)}{S(t)} = r\,dt + \sigma(S(t), t)\,dw^*(t),$$

$$d\xi(t)^{-1} = \frac{\mu(\omega, t) - r}{\sigma(S(t), t)}\xi(t)^{-1}\,dw^*(t).$$

Thus, S is a diffusion process under Q, while ξ^{-1} is a martingale under Q. Invoking conditional expectation under Q, we get

$$E_Q\left[\frac{1}{U'(S(T))}\;\middle|\; S(T) = S\right] = \frac{e^{rT}}{\lambda}\xi(t)^{-1},$$

where the left-hand side must be a function of S and t only, since S is a diffusion process under Q. Define

$$h(S, t) \equiv -\ln\left(E_Q\left[\frac{1}{U'(S(T))}\;\middle|\; S(T) = S\right]\right) + rT - \ln\lambda.$$

Then $\ln \xi(t) = h(S(t), t)$. Assuming h is twice continuously differentiable with respect to S and continuously differentiable with respect to t, we have, by Itô's lemma,

$$\mu(\omega, t) = r - \sigma(S(t), t)^2 S(t) h_S(S(t), t).$$

We conclude that μ is a function of S and t. Consequently, S is a diffusion process.

Third, we have so far assumed that the representative investor does not receive any exogenous income. This is not necessary. Our results hold as long as the exogenous income (to be distributed at the final date) is a function of the current stock price (but not the history of stock prices). In particular, our results hold if the exogenous income is nonrandom, because with such an exogenous income distribution the representative investor's marginal utility or shadow prices is still path independent.

Finally, we mention briefly that we can extend our characterization of equilibrium asset price processes to the case when there is more than one risky stock. The basic technique should be the same as in the case with a single stock, that is, to exploit the condition that the state-price process ξ be path independent in equilibrium. For example, in the case of two risky stocks, suppose that

$$dS_1 = \mu_1 S_1\,dt + \sigma_1 S_1\,dw_1,$$

$$dS_2 = \mu_2 S_2\,dt + \sigma_2 S_2(\rho\,dw_1 + \sqrt{1 - \rho^2}\,dw_2),$$

where μ_i, σ_i, and ρ are functions of S_1, S_2, and t. As in the one-dimensional case, define

$$\begin{pmatrix} f \\ g \end{pmatrix} = (\sigma\sigma^{\mathsf{T}})^{-1}(\mu - r\mathbf{1}),$$

where

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \quad \mathbf{1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \sigma = \begin{pmatrix} \sigma_1 & 0 \\ \sigma_2\rho & \sigma_2\sqrt{1-\rho^2} \end{pmatrix}.$$

Then, in equilibrium, we have

$$S_2 f_2 = S_1 g_1, \tag{15}$$

$$\mathcal{L}f + f_t + \tfrac{1}{2}S_1(\sigma_1^2)_{S_1}(S_1 f_1 + f^2 - f) + \tfrac{1}{2}S_1(\sigma_2^2)_{S_1}(S_2 g_2 + g^2 - g) + S_1(\rho\sigma_1\sigma_2)_{S_1}(fg + S_2 f_2) = 0, \tag{16}$$

$$\mathcal{L}g + g_t + \tfrac{1}{2}S_2(\sigma_1^2)_{S_2}(S_1 f_1 + f^2 - f) + \tfrac{1}{2}S_2(\sigma_2^2)_{S_2}(S_2 g_2 + g^2 - g) + S_2(\rho\sigma_1\sigma_2)_{S_2}(fg + S_1 g_1) = 0, \tag{17}$$

where $\mathcal{L}$ is the differential generator associated with processes S_1 and S_2. The boundary conditions are

$$f(S_1, S_2, T) = -S_1 U''(S_1 + S_2)/U'(S_1 + S_2), \tag{18}$$

$$g(S_1, S_2, T) = -S_2 U''(S_1 + S_2)/U'(S_1 + S_2), \tag{19}$$

for some increasing and concave utility function U and for all $S_1 \geq 0$ and $S_2 \geq 0$. We refer the reader to He and Leland (1992) for a more detailed derivation.

If we require that S be a time-homogeneous diffusion, then (16) and (17) are equivalent to

$$\frac{\sigma_1^2}{2}[S_1 f_1 + f^2 - f] + \rho\sigma_1\sigma_2[S_2 f_2 + fg] + \frac{\sigma_2^2}{2}[S_2 g_2 + g^2 - g] = K \tag{20}$$

for some constant K. This is clearly a generalization of Corollary 1.

3. Asset Price Processes with Constant Volatility

In this section we consider the Black and Scholes economy in which the equilibrium volatility function of stock return is a constant [see Black and Scholes (1973)]. This is an important class of processes, since most stock option pricing models used in practice assume constant volatility. It would be interesting from a theoretical point of view to investigate the dynamics, and in particular the risk premia, of equilibrium price processes when the volatility of stock return is held constant.

If we assume σ is a constant, we can simplify the partial differential equation (7). We summarize our result in the following proposition.

Proposition 1. If the volatility function σ is constant, then a necessary and sufficient condition for $\{S_t, t \in [0, T]\}$ to be an equilibrium in our economy is that the expected instantaneous return μ satisfies the partial differential equation[11]

$$\tfrac{1}{2}\sigma^2 S^2 \mu_{SS} + S\mu\mu_S + \mu_t = 0 \tag{21}$$

with the boundary condition

$$\mu(S, T) = r - \sigma^2 S U''(S)/U'(S) \tag{22}$$

for some increasing and concave utility function U.

Remark 2. (i) Equation (21) implies that $\{\mu(S(t), t), t \in [0, T]\}$ is a local martingale under P. With some additional regularity conditions, μ is a martingale under P. That is, the expected instantaneous returns in the future, given the current stock price, will equal the current expected instantaneous return.

(ii) Similar results hold in the case with multiple stocks. For example, in the case with two stocks, if σ_1, σ_2, and ρ are constants, then (16) and (17) become

$$\mathcal{L}f + f_t = 0, \qquad \mathcal{L}g + g_t = 0; \tag{23}$$

that is, f and g are martingales. This implies that μ_1 and μ_2 are martingales as well, since they are linear combinations of f and g.

Proposition 1 suggests that for any given utility function, it is always possible to find a diffusion process with a constant volatility so that it is an equilibrium process in our economy. To derive the equilibrium expected instantaneous return function for a given level of volatility and a given utility function, we provide two lemmas. The first lemma gives a solution to a heat equation that satisfies a given boundary condition. The proof of this lemma is omitted, since it can be found in standard textbooks. The second lemma utilizes the solution to the heat equation to get a solution to the partial differential equation (PDE) (21) satisfying the boundary condition (22).

[11] After we completed the first version of this article, we became aware of the work of Hodges and Carverhill (1991, revised March 1992), who independently uncovered the same PDE for the equilibrium process of the market in a Black–Scholes economy. Their result is only limited to this special case with a constant volatility.

Lemma 1. Let ϕ be the solution to the heat equation

$$\phi_t = \tfrac{1}{2}\sigma^2 \phi_{xx} \tag{24}$$

with boundary condition

$$\phi(x, 0) = 1/U'(e^x)' \tag{25}$$

where $1/U'(e^x)$ is continuous and bounded above by Ae^{Bx} for some positive constants A and B. Then we have

$$\phi(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{1}{U'(e^{x+\sigma\sqrt{t}z})} e^{-z^2/2}\, dz.$$

To find the solution to the PDE (21), we consider the transformation $V(x, t) = \mu(e^x, T - t) - r$. Since $V_x = \mu_S e^x$, $V_{xx} = \mu_{SS} e^{2x} + \mu_S e^x$, and $V_t = -\mu_t$, (21) becomes

$$\tfrac{1}{2}\sigma^2 V_{xx} + VV_x = V_t + (\tfrac{1}{2}\sigma^2 - r) V_x, \tag{26}$$

and the boundary condition becomes

$$V(x, 0) = -\sigma^2 e^x U''(e^x)/U'(e^x). \tag{27}$$

Next, consider the transformation $W(x, t) = -V(x + (\tfrac{1}{2}\sigma^2 - r)t, t)$. Then (26) becomes

$$\tfrac{1}{2}\sigma^2 W_{xx} = W_t + WW_x \tag{28}$$

with boundary condition

$$W(x, 0) = \sigma^2 e^x U''(e^x)/U'(e^x). \tag{29}$$

Equation (28) is called Burgers' equation [see Kevorkian (1990, p. 31)]. Its solution can be obtained by using the Cole–Hopf transformation: $W(x, t) = -\sigma^2\phi_x(x, t)/\phi(x, t)$, where ϕ is a solution to the heat equation

$$\phi_t = \tfrac{1}{2}\sigma^2\phi_{xx}.$$

The boundary condition for ϕ can be determined by the boundary condition for W, which gives

$$\sigma^2 \frac{\phi_x(x, 0)}{\phi(x, 0)} = -\frac{\sigma^2 e^x U''(e^x)}{U'(e^x)},$$

or, equivalently,

$$\phi(x, 0) = 1/U'(e^x).$$

It is now immediate that

$$\begin{aligned}\mu(S, t) &= r - W\Big(\ln S - \big(\tfrac{1}{2}\sigma^2 - r\big)(T - t),\ T - t\Big)\\ &= r + \sigma^2 \frac{\phi_x\Big(\ln S - \big(\tfrac{1}{2}\sigma^2 - r\big)(T - t),\ T - t\Big)}{\phi\Big(\ln S - \big(\tfrac{1}{2}\sigma^2 - r\big)(T - t),\ T - t\Big)}\end{aligned}$$

is the solution to the PDE (21) satisfying the boundary condition (22). We summarize our disucssion in the following lemma.

Lemma 2. Let ϕ be the solution to the heat equation (24) satisfying the boundary condition (25). Then

$$\mu(S, t) \equiv r + \sigma^2 \frac{\phi_x\Big(\ln S - \big(\tfrac{1}{2}\sigma^2 - r\big)(T - t),\ T - t\Big)}{\phi\Big(\ln S - \big(\tfrac{1}{2}\sigma^2 - r\big)(T - t),\ T - t\Big)}$$

is a solution to the partial differential equation (21) satisfying the boundary condition (22).

We have now obtained the general solution for the expected instantaneous return function corresponding to a given utility function and a constant volatility. Given that our solution of the expected instantaneous return function depends on the length of lifetime horizon, it would be interesting to investigate the dynamic behavior of stock prices for very large T or when T goes to infinity. This is similar to the approach used in the standard turnpike theory for intertemporal consumption and portfolio policies. To illustrate our results, we consider two more examples.

Example 2. Assume $r = 0$. Consider the HARA class of utility functions

$$-U'(S)/U''(S) = A + BS,$$

where $A, B \geq 0$. By integration, we have $U'(S) = K(A + BS)^{-1/B}$, where $K > 0$ is a constant. For simplicity, we take $K = 1$. Hence,

$$\phi(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (A + Be^{x + \sigma\sqrt{t}z})^{1/B} e^{-z^2/2}\, dz.$$

If $B = 1$, $\phi(x, t) = e^x e^{\sigma^2 t/2} + A$. According to Lemma 2, we conclude that

$$\mu(S, t) = \sigma^2(S/(S + A)).$$

Thus, S is a time-homogeneous diffusion process. If $B = \frac{1}{2}$, then $\phi(x, t) = A^2 + Ae^{x + \sigma^2 t/2} + \frac{1}{4} e^{2x + 2\sigma^2 t}$. Hence,

$$\mu(S, t) = \sigma^2 S \frac{A + \frac{1}{2} S e^{\sigma^2 (T - t)}}{A^2 + AS + \frac{1}{4} S^2 e^{\sigma^2 (T - t)}}.$$

Clearly, S is not a time-homogeneous diffusion process. If we let $T \to +\infty$, we get $\mu = 2\sigma^2$. Thus, if the lifetime horizon is sufficiently long, S behaves approximately like a geometric Brownian motion. Moreover, the constant A has no real effect on the price process.

Example 3. Assume $r = 0$. Consider a utility function $U(S) = \ln S + S^{1-\alpha}/(1-\alpha)$, where $\alpha > 0$. We saw in Section 2 that there is no time-homogeneous diffusion process that is consistent with an equilibrium with this utility function. Since $U'(S) = S^{-1} + S^{-\alpha}$, we find

$$\phi(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{1}{e^{-x-\sigma\sqrt{t}z} + e^{-\alpha x-\alpha\sigma\sqrt{t}z}} e^{-z^2/2}\, dz,$$

$$\phi_x(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{e^{-x-\sigma\sqrt{t}z} + \alpha e^{-\alpha x-\alpha\sigma\sqrt{t}z}}{(e^{-x-\sigma\sqrt{t}z} + e^{-\alpha x-\alpha\sigma\sqrt{t}z})^2} e^{-z^2/2}\, dz.$$

According to Lemma 2, we have

$$\mu(S, t) = \sigma^2 \frac{\phi_x\left(\ln S - \frac{1}{2}\sigma^2(T - t)\ T - t\right)}{\phi\left(\ln S - \frac{1}{2}\sigma^2(T - t)\ T - t\right)}.$$

Clearly, S is not a time-homogeneous diffusion process. If we let T go to infinity, then μ converges to σ^2 if $\alpha \leq 1$ and to $\sigma^2\alpha$ if $\alpha > 1$. Thus, if the lifetime horizon is sufficiently long, the equilibrium price will behave approximately like a geometric Brownian motion. Note that when $0 < \alpha < 1$, $\ln S$ dominates $S^{1-\alpha}/(1 - \alpha)$, so the equilibrium price is determined by the log utility. When $\alpha > 1$, $S^{1-\alpha}/(1 - \alpha)$ dominates $\ln S$, and therefore the equilibrium price is determined by $S^{1-\alpha}/(1 - \alpha)$.

The following proposition establishes an important property of the expected instantaneous return function when the utility function of the representative investor exhibits increasing or decreasing relative-risk aversion. We need to impose some regularity conditions for this proposition.

Condition R

1. *There exists a constant $K > 0$ such that for all $x, y > 0$ and $t \in [0, t]$,*

$$|x\mu(x, t) - y\mu(y, t)| \leq K|x - y|, \qquad |x\mu(x, t)| \leq K(1 + x),$$

and there exist constants $L > 0$ and $m > 0$ such that for all $x > 0$ and $t \in [0, T]$,

$$\left|\frac{\partial}{\partial x}(x\mu(x, t))\right| + \left|\frac{\partial^2}{\partial x^2}(x\mu(x, t))\right| \leq L(1 + x^m).$$

2. *$xU''(x)/U'(x)$ is twice continuously differentiable for $x > 0$ and its derivatives satisfy a polynomial growth condition, where U is the utility function for the representative investor.*

Proposition 2. Suppose Condition R is satisfied. Let μ be the equilibrium expected instantaneous return function such that it satisfies the conditions in Proposition 1 with a constant σ. Then, for every $t \in [0, T]$, $\mu(x, t)$ is increasing (decreasing) in x if and only if the relative risk aversion $-xU''(x)/U'(x)$ is increasing (decreasing) in x.

Proof. The necessary part of this proposition is obvious. We only prove the sufficient part. First, we note that given the regularity conditions imposed on μ and the fact that μ satisfies the necessary conditions in Proposition 1, the Feyman–Kac representation implies that μ is a martingale under P [see, for example, Karatzas and Shreve (1988, Theorem 5.7.6)]. Thus, we can express μ as

$$\mu(x, t) = \mathbf{E}[\mu(S(T), T) \mid S(t) = x].$$

Now, let us fix t and x and denote the price process on $[t, T]$ with $S(t) = x$ by $\{S^x(\tau);\ \tau \in [t, T]\}$. Applying Theorem 5.5 of Friedman (1975) or Theorem 1, Chapter 2, of Gilman and Skorohod (1972), we claim that S^x is differentiable with respect to x, the initial data S^x. Furthermore, let D^x denote the derivative of S^x with respect to x defined on $[t, T]$. Then D^x satisfies

$$dD^x(\tau) = (\mu(S^x(\tau), \tau) + S^x(\tau)\mu_S(S^x(\tau), \tau))D^x(\tau)\ d\tau + \sigma D^x(\tau)\ dw(\tau),$$

with $D^x(t) = 1$. Since $|\mu(S, t) + S\mu_S(S, t)|$ is continuous and bounded above by a polynomial function, $\int_t^T |\mu(S^x(\tau), \tau) + S^x(\tau)\mu_S(S^x(\tau), \tau)|\ d\tau < \infty$, P-a.s. It then follows that

$$D^x(s) = \exp\left\{\int_t^s \left(\mu(S^x(\tau), \tau) + S^x(\tau)\mu_S(S^x(\tau), \tau) - \frac{1}{2}\sigma^2\right) d\tau + \sigma(w(s) - w(t))\right\} > 0.$$

Following again Theorem 5.5 of Friedman (1975), we obtain

$$\begin{aligned}\mu_S(x, t) &= \mathbf{E}_t[\mu_S(S^x(T), T)D^x(T)] \\ &= \sigma^2\mathbf{E}_t\left[\frac{\partial}{\partial y}\left(-\frac{yU''(y)}{U'(y)}\right)\bigg|_{y=S^x(T)} D^x(T)\right].\end{aligned}$$

Since $D^x(T) > 0$, we conclude that $\mu_S(S, t)$ is increasing (decreasing) in S for $t \in [0, T]$ if the relative-risk aversion is increasing (decreasing) in terminal wealth. ■

Now, consider the dynamics for the log price,

$$d \ln S(t) = (\mu(S(t), t) - \tfrac{1}{2}\sigma^2)\, dt + \sigma\, dw(t).$$

When μ is decreasing in the level of the stock price, the log-price process always exhibits "mean reversion" in the sense that the return of the stock price moves in the opposite direction to the level of the stock price. Proposition 2 demonstrates that mean reversion is naturally associated with preferences that exhibit decreasing relative-risk aversion, when the volatility of stock return is constant. Similarly, "mean aversion" processes are naturally associated with preferences that exhibit increasing relative-risk aversion.

4. Concluding Remarks

We have provided an approach to characterize equilibrium asset price processes within the family of diffusion processes in a specialized pure exchange economy. In our economy, we assume that the securities markets are dynamically complete, all investors have a state-independent utility function and receive no exogenous income, and all investors consume at some fixed future date. We derive our characterization by exploiting the equilibrium condition that the representative investor optimally holds the total supply of risky assets and therefore follows a path-independent strategy. Consequently, our equilibrium conditions include, but are stronger than, those derived in the intertemporal capital asset pricing models.

Market completeness has played an important role in our analysis. It allows us to formulate the model by using a representative investor framework. It also permits us to determine the unique Arrow–Debreu state prices. It would therefore be interesting to see how our approach can be generalized to an incomplete markets setting.

The assumptions that investors receive no exogenous income and that their preferences can be represented by a von Neumann–Morgenstern, state-independent utility function are also crucial ones. For example, if we had allowed state-dependent utility functions or random exogenous income, then any arbitrage-free asset price process could be supported by an equilibrium. Given that there is a growing interest in non-time-additive utility functions, such as utility functions that exhibit habit formation, it would be interesting to extend our analysis for such utility functions.

Appendix

Proof of Theorem 1. We prove the sufficiency of Theorem 1. To do so, we transform the representative investor's dynamic consumption and investment problem into a static utility maximization problem:

$$\sup_{W(T)\geq 0} \mathbf{E}U(W(T)),$$

$$\mathbf{E}[\xi(T)W(T)e^{-rT}] \leq W_0, \tag{A1}$$

In other words, the dynamic budget constraint in (4) is replaced by a static budget constraint [see Cox and Huang (1989)]. The first-order condition for the static problem is that there exists a scalar $\lambda > 0$ so that

$$U'(W(T)) \begin{cases} = \lambda\xi(T)e^{-rT}, & \text{if } W(T) > 0, \\ \leq \lambda\xi(T)e^{-rT}, & \text{if } W(T) = 0. \end{cases}$$

Since in equilibrium the representative investor holds one unit of the stock and consumes the final value of the stock at the final date, we have $W(T) = S(T)$ in equilibrium. Because we have assumed that $S(T) > 0$, a.s., the first-order condition should always hold in equality.

As we discussed earlier, the utility function that supports this equilibrium satisfies

$$\frac{U'(S)}{U'(S_0)} = \exp\left(-\int_{S_0}^{S} \frac{f(x, T)}{x}\, dx\right),$$

where $S_0 = S(0)$. Define $g(S, t) = -\int_{S_0}^{S} f(x, t)/x\, dx$. It is now sufficient to show that $g(S(T), T) = \ln \xi(T) + c$ for some constant c. To do so, we apply Itô's lemma to g:

$$\begin{aligned} dg(S, t) &= -\frac{f}{S}(\mu S\, dt + \sigma S\, dw) - \left(\frac{f_S}{S} - \frac{f}{S^2}\right)\frac{\sigma^2 S^2}{2}\, dt \\ &\quad - \left(\int_{S_0}^{S} \frac{f_t(x, t)}{x}\, dx\right) dt \\ &= -\frac{\mu - r}{\sigma}\, dw - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 dt + rf\, dt \\ &\quad - \left(\frac{\sigma^2}{2}(f^2 + Sf_S - f) + \int_{S_0}^{S} \frac{f_t(x, t)}{x}\, dx\right) dt \\ &= d\ln\xi(t) - \left(rf + \frac{\sigma^2}{2}(f^2 + Sf_S - f) + \int_{S_0}^{S} \frac{f_t(x, t)}{x}\, dx\right) dt. \end{aligned} \tag{A2}$$

Now, define

$$I(S, t) = rf + (\sigma^2/2)(f^2 + Sf_S - f).$$

Since $I(S, t) = I(S_0, t) + \int_{S_0}^{S} I_S(x, t)\, dx$ and f satisfies the PDE (7), we have

$$I(S, t) = I(S_0, t) + \int_{S_0}^{S} \Big(r f_S(x, t) + \sigma(x, t)\sigma_S(x, t) \times (f^2(x, t) + x f_S(x, t) - f(x, t)) + \frac{\sigma(x, t)^2}{2}(2f(x, t) f_S(x, t) + x f_{SS}(x, t)) \Big) dx$$

$$= I(S_0, t) - \int_{S_0}^{S} \frac{f_t(x, t)}{x}\, dx.$$

Substituting $I(S, t)$ into (A2), we get

$$dg(S(t), t) = d \ln \xi(t) - I(S_0, t)\, dt.$$

Integrating both sides, we find

$$g(S(T), T) = \ln \xi(T) - \int_0^T I(S_0, t)\, dt.$$

This completes our proof. ■

References

Bick, A., 1990, "On Viable Diffusion Price Processes of the Market Portfolio," *Journal of Finance*, 45, 673–689.

Black, F., and M. Scholes, 1973, "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, 81, 637–659.

Breeden, D., 1979, "An Intertemporal Asset Pricing Model with Stochastic Consumption and Investment Opportunities," *Journal of Financial Economics*, 7, 265–296.

Breeden, D., and R. Litzenberger, 1979, "Prices of State-Contingent Claims Implicit in Option Prices," *Journal of Business*, 51, 621–651.

Brennan, M., 1979, "The Pricing of Contingent Claims in Discrete Time Models," *Journal of Finance*, 34, 53–68.

Constantinides, G., 1982, "Intertemporal Asset Pricing with Heterogenous Consumers and without Demand Aggregation," *Journal of Business*, 55, 253–267.

Cox, J., and C. Huang, 1989, "Optimal Consumption and Portfolio Policies When Asset Prices Follow a Diffusion Process," *Journal of Economic Theory*, 49, 33–83.

Cox, J., J. Ingersoll, and S. Ross, 1985, "An Intertemporal General Equilibrium Model of Asset Prices," *Econometrica*, 53, 363–384.

Cox, J., and H. Leland, 1982, "On Dynamic Investment Strategies," *Proceedings of Seminar on the Analysis of Security Prices*, Center for Research in Security Prices (CRSP), Graduate School of Business, University of Chicago, 139–173.

Dybvig, P., and C. Huang, 1988, "Nonnegative Wealth, Absence of Arbitrage, and Feasible Consumption Plans," *Review of Financial Studies*, 1, 377–401.

Friedman, A., 1975, *Stochastic Differential Equations and Applications*, Academic Press, New York.

Friedman, A., 1976, *Partial Differential Equations*, Krieger, New York.

Gennotte, G., and T. Marsh, 1993, "Variations in Economic Uncertainty and Risk Premiums on Capital Assets"; forthcoming in *European Economic Review.*

Gilman, I., and A. Skorohod, 1972, *Stochastic Differential Equations,* Springer-Verlag, New York.

Harrison, M., and D. Kreps, 1979, "Martingale and Multiperiod Securities Markets," *Journal of Economic Theory,* 20, 381–408.

Hansen, L., 1982, "Large Sample Properties of Generalized Method of Moments Estimators," *Econometrica,* 50, 1029–1054.

Harvey, C., 1989, "Time-Varying Conditional Covariances in Tests of Asset Pricing Models," *Journal of Financial Economics,* 24, 289–317.

He, H., and H. Leland, 1992, "Equilibrium Asset Price Processes," working paper, University of California at Berkeley.

Hodges, S., and A. Carverhill, 1991, "The Characterization of Economic Equilibria Which Support Black-Scholes Option Pricing," Financial Options Research Centre, University of Warwick, forthcoming in *Economic Journal.*

Huang, C., 1987, "An Intertemporal General Equilibrium Asset Pricing Model: The Case of Diffusion Information," *Econometrica,* 55, 117–142.

Karatzas, I., and S. Shreve, 1988, *Brownian Motion and Stochastic Calculus,* Springer-Verlag, New York.

Kevorkian, J., 1990, *Partial Differential Equations: Analytical Solution Techniques,* Wadsworth, Belmont, CA.

Kreps, D., 1981, "Arbitrage and Equilibrium in Economies with Infinitely Many Commodities," *Journal of Mathematical Economics,* 8, 15–35.

Merton, R., 1971, "Optimal Consumption and Portfolio Rules in a Continuous Time Model," *Journal of Economic Theory,* 3, 373–413.

Merton, R., 1973, "An Intertemporal Capital Asset Pricing Model," *Econometrica,* 41, 867–887.

Rubinstein, M., 1976, "The Valuation of Uncertain Income Streams and the Pricing of Options," *Bell Journal of Economics,* 7, 407–425.

Wang, S., 1991, "The Integrability Problem of Asset Prices"; forthcoming in *Journal of Economic Theory.*