DOUBLE LOOKBACKS*

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Abstract

A new class of options, *double lookbacks*, where the payoffs depend on the maximum and/or minimum prices of one or two traded assets is introduced and analyzed. This class of double lookbacks includes calls and puts with the underlying being the difference between the maximum and minimum prices of one asset over a certain period, and calls or puts with the underlying being the difference between the maximum prices of two correlated assets over a certain period. Analytical expressions of the joint probability distribution of the maximum and minimum values of two correlated geometric Brownian motions are derived and used in the valuation of double lookbacks. Numerical results are shown, and prices of double lookbacks are compared to those of standard lookbacks on a single asset.

Key Words: Extreme values, correlated Brownian motions, joint density functions, double lookbacks, semi-lookbacks

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1 Introduction

Exotic options designed as contingent claims on equity indices, currency exchange rates, and the term structure of interest rates, have achieved enormous success in global financial markets during the past decade. Exotic derivative products, while seemingly complicated to small investors, have provided institutional investors new vehicles to meet their diverse financial needs, e.g., hedging, risk management, or speculation (when investors have specific views on the market movements in the future). Theoretical advances by academics as well as practitioners have helped market participants create more exotic products and understand the economic benefits of such products, and consequently, contributed in an important way to the surge in popularity of exotic options.

In this paper we introduce a new class of exotic options: lookback options based on two traded assets. A standard one-asset lookback call (or put) gives its holder the right to buy (or sell) the underlying stock at its historical minimum (or maximum) price over a certain period. Analytical solutions for standard lookback options have been found by Goldman, Sosin, and Gatto (1979) and Goldman, Sosin, and Shepp (1979). Lookback options are appealing because they offer investors the opportunity (at a price, of course) of buying a stock at its lowest price and selling a stock at its highest price.

In our two-asset generalization, we consider options whose payoffs depend on the extremal (i.e., maximum and/or minimum) prices of one and/or two stocks over a given period. For example, we consider call or put options on the spread between the maximum and minimum price of Xerox stock over a given interval of time; an option to receive the maximum of General Motors' stock price (or return) at the maximum of Ford's stock price (or return) over a given period; an option to receive the minimum of IBM's stock price (or return) at the minimum of Digital's stock price (or return) over a given period. We refer to these options as *double lookbacks.*

The economic motivation for double lookbacks is not difficult to perceive. An option on the spread between the maximum and minimum price of a single stock over a given interval of time captures in part the idea of an option on price volatility, and is conceptually simpler. Such an option might be of interest to traders who want to bet on price volatility or hedge an existing position which is sensitive to price volatility. Double lookbacks involving two assets allow investors to bet on the difference between the extreme values of two assets or two indices. Since the double lookback is an exchange between two extreme values, it is cheaper and therefore more attractive than a lookback option that exchanges one asset for the extreme value of that asset. If an investor wants to take a long position in the maximum of one asset and a short position in the value of another asset, then a *semi* double lookback, which is an option on the difference between the maximum price of one asset over a given period and the terminal price of another asset, would be an appropriate investment vehicle.

The main contribution of the paper is to derive analytical expressions for the density/distribution functions necessary to price double lookback options. In the case of a single asset, we derive the analytical expression for the probability distribution of the maximum and the minimum prices of one asset following a geometric Brownian motion. In the case of two assets, we derive the analytical expression for the probability distribution of the maximum and/or minimum prices of two assets following two correlated geometric Brownian motions. Numerical procedures are readily available for evaluating all of the double lookback options discussed in the paper.

It is useful to point out that while the focus of this paper is on the pricing of lookback options, the analysis presented in the paper can also be used to value barrier (or knockout) options based on two traded assets. The analytical solution for the one-asset barriers is well known, see Merton (1973) and Rubinstein (1992). For a two-asset example, we may consider an option on the difference of two asset prices subject to a knockout condition based on either one or both assets' prices not reaching the boundary. We refer to these options as *double barriers*.

The paper is organized as follows. In section 2 we present mathematical results that are important for the evaluation of double lookbacks. Specifically, we present various joint density/distribution functions of the extreme values of two correlated Brownian motions. In section 3 we set up the economy and analyze in details various types of lookback options of interest, including lookbacks on one asset, double lookbacks, semi-lookbacks as well as double barriers. Numerical examples are provided and compared. We conclude the paper in Section 4.

2 Main Mathematical Results

In this section we present several mathematical theorems that are necessary for our examination of the no-arbitrage pricing of double lookback options. Specifically, we present the joint probability density/distribution functions of the extreme values (i.e., maximum and minimum) of two correlated Brownian motions. To the best of our knowledge, these joint probability density/distribution functions have never been worked out in the literature. For completeness, we also report the joint probability density/distribution functions of the maximum and minimum of a one-dimensional Brownian motion and the joint probability density/distribution function of the maximum or minimum of one Brownian motion and the end value of another Brownian motion. The proofs for all of the theorems in this section are left in the Appendix.

To start, let us define two Brownian motions with drifts,

$$X_i(t) \equiv \alpha_i t + \sigma_i w_i(t), \qquad t \geq 0$$

where α_i and σ_i are constants, w_i is a Brownian motion defined on some probability space, and $cov(w_1(t), w_2(t)) = \rho t$ with ρ being a constant. Next, define the running minimum and maximum of X_i by

$$\begin{aligned}
\underline{X}_i(t) &= \min_{0 \leq s \leq t} X_i(s) \\
\overline{X}_i(t) &= \max_{0 \leq s \leq t} X_i(s).
\end{aligned}$$

Our first theorem displays the probability density/distribution functions of the maximum or minimum of a one-dimensional Brownian motion with constant drift and the joint probability density/distribution functions of the maximum and minimum of a one-dimensional Brownian motion. For ease of presentations, we define

$$\begin{aligned}
\mathcal{P}(X_1(t) \in dx, \overline{X}_1(t) \leq x_1) &\equiv g(x, x_1, t; \alpha_1)dx, \quad x \leq x_1, x_1 \geq 0 \\
\mathcal{P}(\overline{X}_1(t) \leq x_1) &\equiv G(x_1, t; \alpha_1), \quad x_1 \geq 0 \\
\mathcal{P}(X_1(t) \in dx, \underline{X}_1(t) \geq x_1, \overline{X}_1(t) \leq x_2) &\equiv g_{+-}(x, x_1, x_2, t; \alpha_1)dx, \quad x_1 \leq 0 \leq x_2, x \in [x_1, x_2] \\
\mathcal{P}(\underline{X}_1(t) \geq x_1, \overline{X}_1(t) \leq x_2) &\equiv G_{+-}(x_1, x_2, t; \alpha_1), \quad x_1 \leq 0 \leq x_2, x \in [x_1, x_2],
\end{aligned}$$

By considering the extremes of $-X_1$ instead of X_1 and adjusting the drift from α_1 to α_2, it is not hard to see that

$$\mathcal{P}(X_1(t) \in dx, \underline{X}_1(t) \geq x_1) = g(-x, -x_1, t; -\alpha_1), \quad x \geq x_1, \; x_1 \leq 0$$

$$\mathcal{P}(\underline{X}_1(t) \geq x_1) = G(-x_1, t; -\alpha_1), \quad x_1 \leq 0$$

Theorem 1 *(i) The probability density/distribution functions for the maximum (or the minimum) of a Brownian motion with constant drift is given by:*

$$g(x, x_1, t; \alpha_1) = \frac{1}{\sigma_1 \sqrt{t}} \phi(\frac{x - \alpha_1 t}{\sigma_1 \sqrt{t}}) \left(1 - e^{-\frac{4x_1^2 - 4x_1 x}{2\sigma_1^2 t}}\right), \quad x \leq x_1, \; x_1 \geq 0 \tag{1}$$

$$G(x_1, t; \alpha_1) = N\left(\frac{x_1 - \alpha_1 t}{\sigma_1 \sqrt{t}}\right) - e^{\frac{2\alpha_1 x_1}{\sigma_1^2}} N\left(\frac{-x_1 - \alpha_1 t}{\sigma_1 \sqrt{t}}\right), \quad x_1 \geq 0 \tag{2}$$

where $\phi(z) = \exp(-z^2/2)/\sqrt{2\pi}$ is the standard normal density, and $N(\bullet)$ is the corresponding distribution function.

(ii) The joint probability density function of the maximum, minimum, and end point of a Brownian motion with a constant drift, for $x \in [x_1, x_2]$, $x_1 \le 0$, $x_2 \ge 0$, is given by:

$$g_{+-}(x, x_1, x_2) \tag{3}$$
$$= \exp\left(\frac{\alpha_1 x}{\sigma_1^2} - \frac{\alpha_1^2 t}{2\sigma_1^2}\right) \sum_{n=-\infty}^{\infty} \frac{1}{\sigma_1\sqrt{t}} \left[\phi\left(\frac{x - 2n(x_2 - x_1)}{\sigma_1\sqrt{t}}\right) - \phi\left(\frac{x - 2n(x_2 - x_1) - 2x_1}{\sigma_1\sqrt{t}}\right)\right] dx,$$

This density can be expressed in the equivalent form

$$g_{+-}(x, x_1, x_2) \tag{4}$$
$$= \frac{2}{x_2 - x_1} \exp\left(\frac{\alpha_1}{\sigma_1^2} x - \frac{\alpha_1^2}{2\sigma_1^2} t\right) \sum_{n=1}^{\infty} \exp\left(-\frac{n^2\pi^2\sigma_1^2 t}{2(x_2 - x_1)^2}\right) \sin n\pi\left(\frac{-x_1}{x_2 - x_1}\right) \sin n\pi\left(\frac{x - x_1}{x_2 - x_1}\right) dx.$$

(iii) The joint probability distribution function of the maximum and minimum of a Brownian motion with constant drift, for $x_1 \le 0$, $x_2 \ge 0$, is given by:

$$G_{+-}(x_1, x_2, t; \alpha_1) \tag{5}$$
$$= \sum_{n=-\infty}^{\infty} e^{\frac{2n\alpha_1(x_2 - x_1)}{\sigma_1^2}} \left\{\left[N\left(\frac{x_2 - \alpha_1 t - 2n(x_2 - x_1)}{\sigma_1\sqrt{t}}\right) - N\left(\frac{x_1 - \alpha_1 t - 2n(x_2 - x_1)}{\sigma_1\sqrt{t}}\right)\right]\right.$$
$$\left. - e^{\frac{2x_1\alpha_1}{\sigma_1^2}} \left[N\left(\frac{x_2 - \alpha_1 t - 2n(x_2 - x_1) - 2x_1}{\sigma_1\sqrt{t}}\right) - N\left(\frac{x_1 - \alpha_1 t - 2n(x_2 - x_1) - 2x_1}{\sigma_1\sqrt{t}}\right)\right]\right\}.$$

This distribution can be written in the equivalent form

$$G_{+-}(x_1, x_2, t; \alpha_1) = e^{-\frac{\alpha_1^2 t}{2\sigma_1^2}} \sum_{n=1}^{\infty} \frac{2n\pi\left[e^{\frac{\alpha_1 x_1}{\sigma_1^2}} - (-1)^n e^{\frac{\alpha_1 x_2}{\sigma_1^2}}\right]}{n^2\pi^2 + \left[\frac{\alpha_1(x_2 - x_1)}{\sigma_1^2}\right]^2} e^{-\frac{n^2\pi^2\sigma_1^2 t}{2(x_2 - x_1)^2}} \sin\frac{n\pi(-x_1)}{x_2 - x_1}. \tag{6}$$

Remark 1 *One should note that the two seemingly different formulas in (ii) and (iii) for the joint density/distribution are in fact equal: the second is the Fourier sine transform of the first. In practice, one or the other may lead to more useful numerical approximations depending upon the problem.*

In Figure 1, we present a surface and contour plot of the joint density function

$$-\frac{\partial^2 G_{+-}(x_1, x_2, t; \alpha_1)}{\partial x_1 \partial x_2}.$$

The parameter values chosen are $\sigma_1 = 0.2$ yr$^{-1/2}$, $\alpha_1 = 0$, and $t = 1$ yr, and we plot the minimum over the range $[-0.7, 0]$ and the maximum over the range $[0, 0.7]$. Note that in calculating the infinite sum, we realize that the fact that when $|n|$ is large, the terms in the bracket converges to zero at higher order. Our plot shows that the density function is smooth and well-behaved.

Next, we present our main theorem on the probability density/distribution functions of the extreme values of two correlated Brownian motions. The no-arbitrage pricing of double lookback options is a direct consequence of this theorem. Define

$$\begin{aligned}&\mathcal{P}(X_1(t) \in dx_1, X_2(t) \in dx_2, \underline{X}_1(t) \geq m_1, \underline{X}_2(t) \geq m_2)\\ &\equiv p(x_1, x_2, t; m_1, m_2, \alpha_1, \alpha_2, \sigma_1, \sigma_2, \rho)\, dx_1 dx_2\end{aligned}$$

Theorem 2 *(i) For $x_1 \geq m_1$, $x_2 \geq m_2$, where $m_1 \leq 0$, $m_2 \leq 0$,*

$$\begin{aligned}&p(x_1, x_2, t; m_1, m_2, \alpha_1, \alpha_2, \sigma_1, \sigma_2, \rho)\\ &= \frac{e^{a_1 x_1 + a_2 x_2 + bt}}{\sigma_1 \sigma_2 \sqrt{1-\rho^2}} h(x_1, x_2, t; m_1, m_2, \alpha_1, \alpha_2, \sigma_1, \sigma_2, \rho)\end{aligned} \tag{7}$$

where

$$h(x_1, x_2, t; m_1, m_2, \alpha_1, \alpha_2, \sigma_1, \sigma_2, \rho) = \frac{2}{\beta t} \sum_{n=1}^{\infty} e^{-\frac{r^2 + r_0^2}{2t}} \sin \frac{n\pi\theta_0}{\beta} \sin \frac{n\pi\theta}{\beta} I_{\frac{n\pi}{\beta}} \left(\frac{r r_0}{t} \right)$$

and

$$\begin{aligned}
a_1 &= \frac{\alpha_1 \sigma_2 - \rho \alpha_2 \sigma_1}{(1-\rho^2)\sigma_1^2 \sigma_2}, \quad a_2 = \frac{\alpha_2 \sigma_1 - \rho \alpha_1 \sigma_2}{(1-\rho^2)\sigma_1 \sigma_2^2} \\
b &= -\alpha_1 a_1 - \alpha_2 a_2 + \frac{1}{2}\sigma_1^2 a_1^2 + \rho \sigma_1 \sigma_2 a_1 a_2 + \frac{1}{2}\sigma_2^2 a_2^2 \\
\tan \beta &= -\frac{\sqrt{1-\rho^2}}{\rho}, \quad \beta \in [0, \pi], \\
z_1 &= \frac{1}{\sqrt{1-\rho^2}} \left[\left(\frac{x_1 - m_1}{\sigma_1} \right) - \rho \left(\frac{x_2 - m_2}{\sigma_2} \right) \right], \quad z_2 = \left(\frac{x_2 - m_2}{\sigma_2} \right) \\
z_{10} &= \frac{1}{\sqrt{1-\rho^2}} \left[-\frac{m_1}{\sigma_1} + \frac{\rho m_2}{\sigma_2} \right], \quad z_{20} = -\frac{m_2}{\sigma_2} \\
r &= \sqrt{z_1^2 + z_2^2}, \quad \tan \theta = \frac{z_2}{z_1}, \quad \theta \in [0, \beta], \\
r_0 &= \sqrt{z_{10}^2 + z_{20}^2}, \quad \tan \theta_0 = \frac{z_{20}}{z_{10}}, \quad \theta_0 \in [0, \beta].
\end{aligned}$$

(ii) For $x_1 \geq m_1$, $x_2 \leq M_2$, where $m_1 \leq 0$, $M_2 \geq 0$, we have

$$\begin{aligned}&\mathcal{P}(X_1(t) \in dx_1, X_2(t) \in dx_2, \underline{X}_1(t) \geq m_1, \overline{X}_2(t) \leq M_2)\\ &= p(x_1, -x_2, t; m_1, -M_2, \alpha_1, -\alpha_2, \sigma_1, \sigma_2, -\rho)\, dx_1 dx_2.\end{aligned}$$

(iii) For $x_1 \leq M_1$, $x_2 \leq M_2$, where $M_1 \geq 0$, $M_2 \geq 0$, we have

$$\mathcal{P}(X_1(t) \in dx_1, X_2(t) \in dx_2, \overline{X}_1(t) \leq M_1, \overline{X}_2(t) \leq M_2)$$
$$= p(-x_1, -x_2, t; -M_1, -M_2, -\alpha_1, -\alpha_2, \sigma_1, \sigma_2, \rho)\, dx_1 dx_2.$$

Remark 2 *(i) For $\alpha_1 = \alpha_2 = 0$ and $\sigma_1 = \sigma_2 = 1$, our result coincides with Carslow (1959, p. 279 or p. 379).*

(ii) Integrating over the density functions in the above theorem and applying a change of polar-coordinates, we obtain the distribution function:

$$\mathcal{P}(\overline{X}_1(t) \leq x_1, \overline{X}_2(t) \leq x_2) = e^{a_1 x_1 + a_2 x_2 + bt} f(r', \theta', t)$$

where a_1, a_2 and b are as defined in Theorem 2, and where

$$f(r', \theta', t) \equiv \frac{2}{\alpha' t} \sum_{n=1}^{\infty} \sin(\frac{n\pi\theta'}{\alpha'}) e^{-\frac{r'^2}{2t}} \int_0^{\alpha'} \sin(\frac{n\pi\theta}{\alpha'}) g_n(\theta) d\theta$$

with

$$g_n(\theta) = \int_0^{\infty} r e^{-\frac{r^2}{2t}} e^{-b_1 r \cos(\theta-\alpha) - b_2 r \sin(\theta-\alpha)} I_{\frac{n\pi}{\alpha}}(\frac{rr'}{t}) dr$$

and

$$\begin{aligned}
r' &= \frac{1}{\sqrt{1-\rho^2}} \left(\frac{x_1^2}{\sigma_1^2} - \frac{2\rho x_1 x_2}{\sigma_1 \sigma_2} + \frac{x_2^2}{\sigma_2^2} \right)^{\frac{1}{2}} \\
\theta' &= \theta + \alpha, \quad \text{with } \cos\theta = \frac{x_1}{\sigma_1 r'} \\
\tan\alpha &= \frac{\rho}{\sqrt{1-\rho^2}}, \quad \alpha' = \alpha + \frac{\pi}{2} \\
b_1 &= a_1\sigma_1 + a_2\sigma_2\rho \\
b_2 &= a_2\sigma_2\sqrt{1-\rho^2}
\end{aligned}$$

Similar expressions can be derived for

$$\mathcal{P}(\underline{X}_1(t) \geq x_1, \overline{X}_2(t) \leq x_2)$$

and

$$\mathcal{P}(\underline{X}_1(t) \geq x_1, \underline{X}_2(t) \geq x_2)$$

It is useful to point out that for certain special values of the correlation coefficient ρ, the density/distribution functions in Theorem 2 can be significantly simplified. We state this result as a corollary.

Corollary 1 *(i) Suppose the same assumptions hold as in Theorem 2 (i), except that the correlation ρ can take on only the special values*

$$\rho_n = -\cos\left(\frac{\pi}{n}\right), \quad n = 1, 2, \ldots.$$

Then the density function p has the special form

$$p(x_1, x_2, t; \bullet) = \frac{e^{a_1 x_1 + a_2 x_2 + bt}}{\sigma_1 \sigma_2 \sqrt{1 - \rho_n^2}} h(z_1, z_2, t; \bullet), \tag{8}$$

where h is a finite *sum of bivariate normal densities*

$$h(z_1, z_1, t; \bullet) = \sum_{k=0}^{n-1} \left[g_k^+(z_1, z_2, t) + g_k^-(z_1, z_2, t) \right].$$

and

$$g_k^{\pm}(z_1, z_2, t) = \pm(2\pi)^{-1} \exp\left(-\frac{1}{2} \left[\left(z_1 - r_0 \cos\left(\frac{2k\pi}{n} \pm \theta_0 \right) \right)^2 + \left(z_2 - r_0 \sin\left(\frac{2k\pi}{n} \pm \theta_0 \right) \right)^2 \right] \right).$$

Similar density functions can be derived for the cases in Theorem 2 (ii) and (iii).
(ii) Suppose the same assumptions hold as in Theorem 2 (ii) with $\rho_n = \cos\frac{\pi}{n}$. Then, for $x_1 \geq 0$, $x_2 \leq 0$,

$$\mathcal{P}(\overline{X}_1(t) < x_1, \underline{X}_2(t) > x_2) = \sum_{k=0}^{n-1} \left[H(r_0, \frac{2k\pi}{n} + \theta) - H(r_0, \frac{2k\pi}{n} - \theta) \right] \tag{9}$$

where

$$\begin{aligned} H(r_0, \frac{2k\pi}{n} \pm \theta) &= \exp\left[A_0 + A_1 \zeta_1 + A_2 \zeta_2 + \left(\frac{1}{2} A_1^2 + \frac{1}{2} A_2^2 + b \right) t \right] \times \\ & \quad N_2 \left(\frac{\zeta_2 + A_2 t}{\sqrt{t}}, \frac{-\sqrt{1 - \rho_n^2}(\zeta_1 + A_1 t) - \rho_n(\zeta_2 + A_2 t)}{\sqrt{t}}, -\rho_n \right) \\ \zeta_1 &= -r_0 \cos(\frac{2k\pi}{n} \pm \theta), \qquad \zeta_2 = r_0 \sin(\frac{2k\pi}{n} \pm \theta) \\ \tan\theta &= \frac{\sqrt{1 - \rho_n^2}}{\rho_n - (x_1 \sigma_2 / x_2 \sigma_1)} \end{aligned}$$

and

$$\begin{aligned} A_0 &= a_1 x_1 + a_2 x_2 \\ A_1 &= a_1 \sigma_1 \sqrt{1 - \rho_n^2} \\ A_2 &= a_1 \sigma_1 \rho_n + a_2 \sigma_2 \end{aligned}$$

and $N_2(x, y, \rho)$ is the standard bivariate normal distribution for correlation ρ.

Remark 3 *Note that the special correlation values in Corollary 1 (i) are all negative. Thus for the double minima or double maxima densities, the result may be of limited usefulness, since we expect most assets to be positively correlated. For the density involving the minimum of one asset and the maximum of another, this corollary is more interesting, since it is applicable to assets with* positive *correlations of the form* $\cos(\pi/n)$ *(where* $n = 3, 4, 5, \cdots$*).*

In Figure 2, we present a surface and contour plot of the joint density function

$$-\frac{\partial^2}{\partial x_1 \partial x_2} \mathcal{P}(\overline{X}_1(t) < x_1, \underline{X}_2(t) > x_2),$$

as given in Corollary 1. The parameter values chosen for the picture are $\sigma_1 = \sigma_2 = 0.2$ yr$^{-1/2}$, $t = 1$ yr, $\alpha_1 = \alpha_2 = 0$, and $\rho = 0.5$. Since this example corresponds to the case where $n = 3$, we can plot the density function using the analytical formula.

Finally, we state a theorem which derives the joint probability density of the extreme value of one Brownian motion and the end points of both Brownian motions. This theorem will be useful for our study of semi-lookback options. Define

$$\mathcal{P}(\overline{X}_1(t) \le M_1, X_1(t) \in dx_1, X_2(t) \in dx_2) \equiv u_{+0}(x_1, x_2, t; M_1, \alpha_1, \alpha_2, \sigma_1, \sigma_2, \rho)\, dx_1\, dx_2$$

$$\mathcal{P}(X_1(t) \in dx_1, X_2 \in dx_2, \underline{X}_2(t) \ge m_2) \equiv u_{0-}(x_1, x_2, t; m_2, \alpha_1, \alpha_2, \sigma_1, \sigma_2, \rho)\, dx_1\, dx_2$$

$$\mathcal{P}(\overline{X}_1(t) \in dx_1, X_2(t) \in dx_2) \equiv f_{+0}(x_1, x_2, t; \alpha_1, \alpha_2, \sigma_1, \sigma_2, \rho)\, dx_1\, dx_2$$

$$\mathcal{P}(X_1(t) \in dx_1, \underline{X}_2(t) \in dx_2) \equiv f_{0-}(x_1, x_2, t; \alpha_1, \alpha_2, \sigma_1, \sigma_2, \rho)\, dx_1\, dx_2$$

Theorem 3 *(1) The probability density function of the maximum of one Brownian motion and the end points of both Brownian motions is given by:*

$$u_{+0}(x_1, x_2, t; \bullet) = \frac{1}{\sigma\sqrt{t}} \phi(\frac{x_1 - \alpha_1 t}{\sigma_1 \sqrt{t}}) \left(1 - e^{-\frac{4M_1^2 - 4M_1 x_1}{2\sigma_1^2 t}}\right) \phi\left(\frac{x_2 - \alpha_2 t - \rho\sigma_2(\frac{x_1 - \alpha_1 t}{\sigma_1})}{\sigma_2\sqrt{(1-\rho^2)t}}\right) \tag{10}$$

Similarly,

$$u_{-0}(x_1, x_2, t; m_2, \alpha_1, \alpha_2, \sigma_1, \sigma_2, \rho) = u_{+0}(-x_2, x_1, t; -m_2, -\alpha_2, \alpha_1, \sigma_2, \sigma_1, -\rho)$$

(2) The probability density function of the maximum of one Brownian motion and the end point of another Brownian motion is given by:

$$f_{+0}(x_1, x_2, t; \bullet) = \frac{1}{\sigma_1 \sigma_2 \pi t^2} \exp{(A)} \left[t\sqrt{1-\rho^2} \exp{(-\frac{B^2}{2})} - d\sqrt{2\pi t} N(-B) \right] \tag{11}$$

with constants

$$
\begin{aligned}
A &= \frac{2\alpha_1 x_1}{\sigma_1^2} - \frac{\alpha_1^2 t}{2\sigma_1^2} - \frac{c^2\rho^2}{2t(1-\rho^2)} + \frac{d^2}{2t(1-\rho^2)} \\
B &= \frac{1}{\sqrt{t(1-\rho^2)}}\left(\frac{x_1}{\sigma_1} + \frac{\alpha_1 t}{\sigma_1}(1-\rho^2) + c\rho^2\right)
\end{aligned}
$$

where

$$
\begin{aligned}
c &= \frac{x_2 - \alpha_2 t}{\sigma_2\rho} - \frac{2x_1 - \alpha_1 t}{\sigma_1} \\
d &= \frac{\alpha_1 t(1-\rho^2)}{\sigma_1} + c\rho^2
\end{aligned}
$$

Similarly,

$$
f_{0-}(x_1, x_2, t; \alpha_1, \alpha_2, \sigma_1, \sigma_2, \rho) = f_{+0}(-x_2, x_1, t; -\alpha_2, \alpha_1, \sigma_2, \sigma_1, -\rho).
$$

In Figure 3, we plot the semi-lookback density f_{+0} for a specific choice of parameter values. Once again, the density function is smooth and well-behaved.

3 No-Arbitrage Valuation of Lookbacks

In this section we apply the mathematical results obtained in the previous section to derive the no-arbitrage valuation of lookback options. We start with the setup of the economy and the basic equation for the no-arbitrage valuation of general contingent claims. For completeness, we include a section on lookback options on a single asset. We then move on to lookback options on two assets, i.e., double lookbacks. For comparison, we also consider semi-lookback options on two assets, i.e., lookbacks on one asset with the payoff depending upon the terminal value of another asset as well. Finally, we discuss how we might be able to use our results to study double barriers, i.e., barrier options on two assets.

3.1 The Economy and Arbitrage-free Pricing

Taken as given is a Black-Scholes economy in which stock prices are log-normally distributed, the interest rate is constant, and continuous trading without transaction costs, taxes, or other market frictions is permitted. There are three assets: one riskfree bond and two risky stocks. The prices of the bond and the stocks are determined by:

$$
\begin{aligned}
B(t) &= e^{rt} \\
S_1(t) &= S_1(0)e^{(\mu_1 - q_1 - \frac{\sigma_1^2}{2})t + \sigma_1 w_1(t)} \\
S_2(t) &= S_2(0)e^{(\mu_2 - q_2 - \frac{\sigma_2^2}{2})t + \sigma_2 w_2(t)}
\end{aligned}
$$

where r is the riskless rate, μ_i the expected instantaneous return of stock i, q_i the dividend yield of stock i, σ_i the volatility of stock i, and w_i a standard Brownian motion with $\text{cov}(dw_1, dw_2) = \rho dt$.

Following Harrison and Kreps (1979) and Harrison and Pliska (1981), the Black-Scholes economy is known to be viable and dynamically complete. Thus, any contingent claim written on these two stocks can be replicated through dynamic trading in the stock(s) and bond. Furthermore, there exists a probability measure Q (the *equivalent martingale* or *risk-neutral* measure) under which the discounted price $V^*(t) = V(t)/B(t)$ of the contingent claim is a martingale. Under this risk neutral probability, the stock price processes are:

$$\begin{aligned} S_1(t) &= S_1(0)e^{(r-q_1-\frac{\sigma_1^2}{2})t+\sigma_1 w_1^*(t)} \\ S_2(t) &= S_2(0)e^{(r-q_2-\frac{\sigma_2^2}{2})t+\sigma_2 w_2^*(t)} \end{aligned}$$

where w_1^* and w_2^* are standard Brownian motions under Q with the same constant correlation ρ as under the original probability measure. Because the discounted contingent claim price $V^*(t)$ is a martingale, its value at date 0 can be determined by taking the conditional expectation of its terminal value:

$$V^*(0) = \mathrm{E}^Q[V^*(T)].$$

For a Black-Scholes economy with constant interest rates, this equation can be rewritten as

$$V(0) = e^{-rT}\mathrm{E}^Q[V(T)].$$

Throughout this paper, we assume that we are pricing the lookback options at date 0, the options expire at date T, and that the lookback period runs from t^* to T. Note that t^* may be either positive or negative. For $i = 1, 2$ and $t \geq t^*$, define the running minimum and maximum of stock price S_i by

$$\begin{aligned} \underline{S}_i(t) &= \min_{t^* \leq s \leq t} S_i(s) \\ \overline{S}_i(t) &= \max_{t^* \leq s \leq t} S_i(s). \end{aligned}$$

A general lookback option can therefore be defined as a contingent claim which pays off at the expiration date T:

$$H(S_1(T), \underline{S}_1(T), \overline{S}_1(T), S_2(T), \underline{S}_2(T), \overline{S}_2(T))$$

where H is a continuous function. The valuation problem we face consists of evaluating a conditional expectation of the payoff function, which in turn is simply a matter of integrating the payout with respect to an appropriate density function.

If the lookback period begins at $t^* > 0$, then by iterating the expectation we find

$$V(0) = e^{-rT}\mathrm{E}^Q\left[\mathrm{E}^Q_{t^*}[V(T)]\right].$$

The inner expectation is of the same form as when $t^* \le 0$, and hence can be evaluated using the same densities derived in the previous section. The outer expectation is simply an integral with respect to the distribution of $S_i(t^*)$, and can be evaluated using standard techniques. For the rest of this paper, we concentrate on the case where $t^* \le 0$.

For ease of notation, we define $X_i(t)$, the normalized logarithmic price of stock i, by

$$X_i(t) = \log S_i(t)/S_i(0) = \alpha_i t + \sigma_i w_i^*(t), \qquad (t^* \le 0 \le t)$$

where $\alpha_i = r - q_i - \sigma_i^2/2$. Also, define the running minimum and maximum of X_i by

$$\begin{aligned}
m_i &= \min_{t^* \le s \le 0} X_i(s) \\
M_i &= \max_{t^* \le s \le 0} X_i(s) \\
\underline{X}_i(t) &= \min_{0 \le s \le t} X_i(s) \\
\overline{X}_i(t) &= \max_{0 \le s \le t} X_i(s).
\end{aligned}$$

Then, we can rewrite the payoff function as

$$H(S_1(0)e^{X_1(T)}, S_1(0)e^{\min(m_1,\underline{X}_1(T))}, S_1(0)e^{\max(M_1,\overline{X}_1(T))}, \cdots, \cdots, \cdots)$$

3.2 Lookbacks on One Asset

In this section we consider lookback options on a single stock. We first record as a proposition the analytical formulas for standard lookbacks based on either the maximum or the minimum of an asset over a given period. The proofs of these formulas can be found in Conze and Viswanathan (1991). We then examine lookback spread options where the payoffs depend upon the maximum and minimum of an asset whose price follows a geometric Brownian motion.

Proposition 1 *Let C_{LB} (or P_{LB}) be the price at time 0 of a standard lookback call (or put) which pays [$S_1(T) - \underline{S}_1(T)$] (or [$\overline{S}_1(T) - S_1(T)$]) at the expiration date T. Then,*

$$\begin{aligned}
C_{LB} &= Se^{-qT}N(d_1) - e^{-rT}mN(d_1 - \sigma\sqrt{T}) \\
&\quad + e^{-rT}\frac{\sigma^2}{2(r-q)}S^{1-\frac{2(r-q)}{\sigma^2}}m^{\frac{2(r-q)}{\sigma^2}}N(-d_1 + \frac{2(r-q)}{\sigma}\sqrt{T}) - \frac{\sigma^2}{2(r-q)}Se^{-qT}N(-d_1) \\
P_{LB} &= -Se^{-qT}N(-d_2) + e^{-rT}MN(-d_2 + \sigma\sqrt{T}) \\
&\quad - e^{-rT}\frac{\sigma^2}{2(r-q)}S^{1-\frac{2(r-q)}{\sigma^2}}M^{\frac{2(r-q)}{\sigma^2}}N(d_2 - \frac{2(r-q)}{\sigma}\sqrt{T}) + \frac{\sigma^2}{2(r-q)}Se^{-qT}N(d_2)
\end{aligned}$$

where $S = S_1(0)$, $m = \underline{S}_1(0)$, $M = \overline{S}_1(0)$ *and*

$$\begin{aligned} d_1 &= \frac{\ln(S/m) + (r-q)T + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \\ d_2 &= \frac{\ln(S/M) + (r-q)T + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \end{aligned}$$

Similarly, let C_M *(or* P_m*) be the price at time* 0 *of a call written on the maximum (or a put on the minimum) with a strike* K*. Then,*

$$C_M = \begin{cases} Se^{-qT}N(d) - e^{-rT}KN(d-\sigma\sqrt{T}) \\ -e^{-rT}\frac{\sigma^2}{2(r-q)}S^{1-\frac{2(r-q)}{\sigma^2}}K^{\frac{2(r-q)}{\sigma^2}}N(d - \frac{2(r-q)}{\sigma}\sqrt{T}) + \frac{\sigma^2}{2(r-q)}Se^{-qT}N(d), & \text{if } K > M \\ e^{-rT}(M-K) + Se^{-qT}N(d_2) - e^{-rT}MN(d_2 - \sigma\sqrt{T}) \\ -e^{-rT}\frac{\sigma^2}{2(r-q)}S^{1-\frac{2(r-q)}{\sigma^2}}M^{\frac{2(r-q)}{\sigma^2}}N(d_2 - \frac{2(r-q)}{\sigma}\sqrt{T}) + \frac{\sigma^2}{2(r-q)}Se^{-qT}N(d_2), & \text{if } K < M \end{cases}$$

$$P_m = \begin{cases} -Se^{-qT}N(-d) + e^{-rT}KN(-d+\sigma\sqrt{T}) \\ +e^{-rT}\frac{\sigma^2}{2(r-q)}S^{1-\frac{2(r-q)}{\sigma^2}}K^{\frac{2(r-q)}{\sigma^2}}N(-d + \frac{2(r-q)}{\sigma}\sqrt{T}) - \frac{\sigma^2}{2(r-q)}Se^{-qT}N(-d), & \text{if } K < m \\ e^{-rT}(K-m) - Se^{-qT}N(-d_1) + e^{-rT}mN(-d_1 + \sigma\sqrt{T}) \\ +e^{-rT}\frac{\sigma^2}{2(r-q)}S^{1-\frac{2(r-q)}{\sigma^2}}m^{\frac{2(r-q)}{\sigma^2}}N(-d_1 + \frac{2(r-q)}{\sigma}\sqrt{T}) - \frac{\sigma^2}{2(r-q)}Se^{-qT}N(-d_1), & \text{if } K > m \end{cases}$$

where

$$d = \frac{\ln(S/K) + (r-q)T + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$

We note that standard lookback call and put prices are equal to the corresponding Black-Scholes values, with strikes set at the current maximum or minimum, plus a premium. This premium reflects the opportunity that the minimum (or maximum) can go down (or up) further. The prices for calls on the minimum and puts on the maximum have a similar interpretation.

Next we consider a *lookback spread* defined as a European call/put option on the spread between the maximum and minimum of a single stock price. Using our notations, the payoff function of such options can be written as

$$\max\Big[0, \Big(\overline{S}_1(T) - \underline{S}_1(T)\Big) - K\Big]$$

$$\max\Big[0, K - \Big(\overline{S}_1(T) - \underline{S}_1(T)\Big)\Big]$$

Lookback spread options are options contingent upon the maximum dispersion of the stock prices realized over a given time horizon. These options are obviously very sensitive to the volatility of the stock prices, and therefore offer tremendous exposure to the volatility of the underlying stock.

Recall that in Theorem 1, we have derived a set of joint density/distribution functions for the maximum and minimum of a single Brownian motion:

$$G_{+-}(x_1, x_2) \equiv \mathcal{P}(\underline{X}_1(t) \geq x_1, \overline{X}_1(s) \leq x_2)$$

The density function is determined by differentiating G_{+-}. Given the joint density function, the price of a lookback spread can be obtained by integrating the final payoffs with respect to the density. Specifically, define

$$V_{SP}(x_1, x_2) = \max\left[0, \left(S_1(0)e^{\max(M_1, x_2)} - S_1(0)e^{\min(m_1, x_1)}\right) - K\right].$$

Proposition 2 *The price at time* 0 *of a lookback spread call,* C_{SP}*, is given by*

$$C_{SP} = e^{-rT} \int_{-\infty}^{0} dx_1 \int_{0}^{\infty} dx_2\, V_{SP}(x_1, x_2) \frac{-\partial^2 G_{+-}(x_1, x_2)}{\partial x_1 \partial x_2} \tag{12}$$

with corresponding formulas for the put option.

Based on (12), we have evaluated the price of a lookback spread option by direct numerical quadrature. In Table 1, we list the prices for lookback spread call and put options for various parameter values.[1] These prices have the right sensitivities, i.e., higher volatility leads to higher option premiums and larger strike price leads to smaller call premium but larger put premium. When the volatility is 20%, the premium for the lookback spread ($\overline{S} - \underline{S}$) is 31.5% of the initial stock price.

3.3 Lookbacks on Two Assets

We now analyze double lookback options whose payoffs depend on the extreme values of two assets following correlated geometric Brownian motions. We are mainly interested in three types of double lookbacks:

- (*Double Maxima*) call or put on the difference between the maximum of S_1 and the maximum of S_2:

$$\max\left[0, \left(a\overline{S}_1(T) - b\overline{S}_2(T)\right) - K\right]$$
$$\max\left[0, K - \left(a\overline{S}_1(T) - b\overline{S}_2(T)\right)\right]$$

  where $a > 0$ and $b > 0$ are parameters to be chosen by investors. In practice, if $t^* = 0$, it may make sense to pick a and b such that $aS_1(0) = bS_2(0)$. For example, $a = \frac{1}{S_1(0)}$ and $b = \frac{1}{S_2(0)}$.

[1] While the density is in the form of infinite sum, it converges rather quickly. We expect that the density calculation is correct to a very high precision almost everywhere, since we sum terms until additional terms contributed less than some very small number. The numbers shown in the table are accurate to the number of digits given.

σ	K	Call	Put
	0	16.21	0.00
0.1	15	3.11	1.17
	20	1.27	4.08
	30	0.18	12.50
	0	31.50	0.00
0.2	30	5.57	2.60
	35	3.56	5.35
	60	0.42	25.99
	0	47.09	0.00
0.3	45	8.46	4.18
	50	6.42	6.89
	100	0.60	48.63

Table 1: Values for call and put lookback spread options with payouts of, respectively, $(\overline{S}-\underline{S}-K)^+$ and $(K-\overline{S}+\underline{S})^+$. We assume that $S(0)=\overline{S}(0)=\underline{S}(0)=100$, $r=0.05$ yr^{-1}, $q=0$, $t=1$ yr.

With a and b chosen in this way, the double maxima represents an option on the difference between the maxmum returns of the two stocks over a given period. When $K=0$, the double maxima call is equivalent to an option to buy the maximum of S_1 at the maximum of S_2.

- (*Double Minima*) call or put on the difference between the minimum of S_1 and the minimum of S_2:

$$\max\Big[0,(a\underline{S}_1(T)-b\underline{S}_2(T))-K\Big]$$
$$\max\Big[0,K-(a\underline{S}_1(T)-b\underline{S}_2(T))\Big]$$

  When $K=0$, the double minima call is equivalent to an option to sell the minimum of S_1 for the minimum of S_2.

- (*Double Lookback Spread*) call or put on the spread between the maximum S_1 and the minimum of S_2:

$$\max\left[0,\left(a\overline{S}_1(T)-b\underline{S}_2(T)\right)-K\right]$$
$$\max\left[0,K-\left(a\overline{S}_1(T)-b\underline{S}_2(T)\right)\right].$$

These three types of options can be useful for investors with special investment needs. Double maxima/minima represent options on the difference between the maximum/minimum returns of two stocks over a given period. They provide investors with a special vehicle to take a view on how these two stocks will perform relative to each other. Similarly, double lookback spreads capture the difference between the maximum upside of one stock and the maximum downside of another stock. This type of options can be an aggressive play on the volatilities of the two stocks as well as on the correlation of the two stocks.

We state a proposition to summarize the evaluation of various double lookback options shown above. Define

$$
\begin{aligned}
V_{Dmax}(x_1, x_2) &= \max\Big[0, aS_1(0)e^{\max(M_1,x_1)} - bS_2(0)e^{\max(M_2,x_2)} - K\Big] \\
V_{Dmin}(x_1, x_2) &= \max\Big[0, aS_1(0)e^{\min(m_1,x_1)} - bS_2(0)e^{\min(m_2,x_2)} - K\Big] \\
V_{DLS}(x_1, x_2) &= \max\Big[0, aS_1(0)e^{\max(M_1,x_1)} - bS_2(0)e^{\min(m_1,x_2)} - K\Big]
\end{aligned}
$$

for some constants a and b.

Proposition 3 *The call prices C_{Dmax}, C_{Dmin} and C_{DLS}, respectively, for double maxima, double minima and double lookback spread options are determined as follows,*

$$
C_{Dmax} = e^{-rT} \int_0^\infty dx_1 \int_0^\infty dx_2\, V_{Dmax}(x_1, x_2) \frac{\partial^2 \mathcal{P}(\overline{X}_1(t) \le x_1, \overline{X}_2(t) \le x_2)}{\partial x_1 \partial x_2} \tag{13}
$$

$$
C_{Dmin} = e^{-rT} \int_{-\infty}^0 dx_1 \int_{-\infty}^0 dx_2\, V_{Dmin}(x_1, x_2) \frac{\partial^2 \mathcal{P}(\underline{X}_1(t) \ge x_1, \underline{X}_2(t) \ge x_2)}{\partial x_1 \partial x_2} \tag{14}
$$

$$
C_{DLS} = e^{-rT} \int_{-\infty}^0 dx_1 \int_0^\infty dx_2\, V_{DLS}(x_1, x_2) \frac{-\partial^2 \mathcal{P}(\underline{X}_1(t) \ge x_1, \overline{X}_2(t) \le x_2)}{\partial x_1 \partial x_2} \tag{15}
$$

Formulas for corresponding puts can be obtained similarly.

Recall that the joint probability distributions for the above three cases have been derived in Theorem 2. This proposition states that the prices of double lookbacks can be obtained by integrating their final payoffs with respect to the corresponding distribution function. In general, evaluation will require a four-dimensional numerical quadrature (a two-dimensional integration is required to get $\mathcal{P}(\underline{X}_1(t) \ge x_1, \overline{X}_2(t) \le x_2)$, given the density function $p(x_1, x_2, t; m_1, m_2, \bullet)$. There exist standard numerical techniques for evaluating such integrals. We expect that this methodology may be more computationally efficient than Monte-Carlo or lattice techniques.[2]

In Table 2, we give numerical prices for various parameter values.[3] Once again, these prices have the desired sensitivities with respect to σ_1, σ_2 and K. We note that as the correlation between the two assets increases, the option premiums decrease for both calls and puts. This should be intuitive, as higher correlation leads to lower volatility in $\overline{S}_1 - \underline{S}_2$. Also, if we hold $\sigma_1 = \sigma_2 = 0.2$,

[2] While we have not conducted a thorough comparison between numerical integration of our function and Monte-Carlo simulations, our preliminary investigation shows that Monte-carlo simulations (without some clever tricks) seem to be distinctly slower for double lookbacks with continuous sampling. However, we believe that for discrete maxima, Monte-Carlo is probably the preferred methodology.

[3] Since we have chosen special ρ's for the cases shown in the table, exact analytical expressions in Corollary 1 were used for computation. The numbers shown should be accurate up to the number of digits given in the table.

σ_1	σ_2	ρ	K	Call	Put
0.2	0.2	any	0	31.50	0.00
		.50	30	7.14	4.17
		.71	30	6.55	3.58
		.90	30	5.92	2.95
0.2	0.4	any	0	44.25	0.00
		.50	45	7.59	6.15
		.71	45	6.69	5.26
		.90	45	5.73	4.28
0.4	0.2	any	0	50.10	0.00
		.50	50	13.28	10.74
		.71	50	12.63	10.14
		.90	50	12.09	9.57
0.4	0.4	any	0	62.84	0.00
		.50	65	12.58	11.57
		.71	65	11.43	10.46
		.90	65	10.32	9.35

Table 2: Values for call and put double lookback spread options with payouts of, respectively, $(\overline{S}_1 - \underline{S}_2 - K)^+$ and $(K - \overline{S}_1 + \underline{S}_2)^+$. We assume that $S_1(0) = S_2(0) = \overline{S}_1(0) = \underline{S}_2(0) = 100$, $r = 0.05\ \mathrm{yr}^{-1}$, $q_1 = q_2 = 0$, $t = 1$ yr. Note that the correlation coefficients correspond to $\rho = \cos\frac{\pi}{3}$, $\cos\frac{\pi}{4}$, and $\cos\frac{\pi}{7}$.

then the option premiums for the double lookback spread in Table 2 is more expensive than those of the lookback spread in Table 1 (for fixed strike). But, as ρ increases to 1, the option premiums converge to those in Table 1.

3.4 Semi-Lookbacks on Two Assets

To make an interesting comparison of various types of lookback options, we add yet another class of lookback options whose payoffs depend on the extreme value of one asset and the final value of another asset. For example, consider an option to buy the maximum of S_1 at the final value of S_2 or an option to sell the minimum of S_2 at the final value of S_1. We call such options *semi-lookbacks.*

Recall that in Theorem 2, we have derived the probability density functions necessary for the valuation of semi-lookbacks:

$$\mathcal{P}(\overline{X}_1(t) \leq M_1, X_1(t) \in dx_1, X_2(t) \in dx_2) = u_{+0}(x_1, x_2, t; \bullet)\, dx_1\, dx_2$$

$$\mathcal{P}(X_1(t) \in dx_1, X_2(t) \in dx_2, \underline{X}_2(t) \geq m_2) = u_{0-}(x_1, x_2, t; \bullet)\, dx_1\, dx_2$$

$$\mathcal{P}(\overline{X}_1(t) \in dx_1, X_2(t) \in dx_2) = f_{+0}(x_1, x_2, t; \bullet)\, dx_1\, dx_2$$

$$\mathcal{P}(X_1(t) \in dx_1, \underline{X}_2(t) \in dx_2) = f_{0-}(x_1, x_2, t; \bullet)\, dx_1\, dx_2$$

For example, consider the call options with the following payoffs:

$$\begin{aligned} V_{Smax}(x_1, x_2) &= \max\Big[0, S_1(0)e^{\max(M_1, x_1)} - S_2(0)e^{x_2} - K\Big] \\ V_{Smin}(x_1, x_2) &= \max\Big[0, S_1(0)e^{x_1} - S_2(0)e^{\min(m_2, x_2)} - K\Big] \end{aligned}$$

with corresponding formulas for the put options. The values of these options are given by the following proposition:

Proposition 4 *Let C_{Smax} and C_{Smin} be the prices at time 0 of semi-lookback call options with payoffs $\Big[\overline{S}_1(T) - S_2(T) - K\Big]^+$ and $[S_1(T) - \underline{S}_2(T) - K]^+$, respectively. Then,*

$$C_{Smax} = e^{-rT} \int_0^\infty dx_1 \int_{-\infty}^\infty dx_2\, V_{Smax}(x_1, x_2)\, f_{+0}(x_1, x_2, T) \tag{16}$$

$$C_{Smin} = e^{-rT} \int_{-\infty}^{+\infty} dx_1 \int_{-\infty}^0 dx_2\, V_{Smin}(x_1, x_2) f_{0-}(x_1, x_2, T). \tag{17}$$

In Table 3, we list values of semi-lookback spread options for various parameter values, obtained by numerical quadrature of the integrals in Proposition 4. Comparing Tables 2 and 3, we find that for a fixed set of volatilities, correlation, and strike parameters, the call premiums for semi-lookback spread options are cheaper than those of the double lookback options. However, the put premiums should be more expensive. As the correlation between two assets increases, the option premiums decrease for both calls and puts.

3.5 Double Barriers

The analysis presented so far can be easily employed for pricing barrier options (or knockouts) based on two assets. In the case with standard one-asset barriers, the option pays $\max(S_1(T) - K, 0)$, subject to a condition that the stock price $S_1(t)$ never hits a fixed boundary (which could either be larger or smaller than the initial stock price). Merton (1973) was the first to provide a solution to evaluate this type of options. We refer the reader to Rubinstein (1992) for a complete list of one-asset barrier options.

In the case with two traded assets, we consider a general contingent claim which pays, at the maturity date,

$$V(T) = f(S_1(T), S_2(T))$$

for some function f, subject to the conditions that $S_1(t)$ and/or $S_2(t)$ never hit some pre-determined boundaries. We shall call such options as *double barriers*. Obviously, the keys to valuing double barriers lie in the following probability density functions, which are derived in Theorem 2:

$$\mathcal{P}(X_1(T) \in dx_1, X_2(T) \in dx_2, \overline{X}_1 \leq K_1, \overline{X}_2 \leq K_2)$$

σ_1	σ_2	ρ	K	Call	Put
0.2	0.2	0.1	0	18.44	4.16
		0.1	15	9.52	9.51
		0.5	0	17.04	2.74
		0.5	15	7.64	7.62
		0.9	0	15.09	0.78
		0.9	15	4.97	4.95
0.2	0.4	0.1	0	25.46	11.16
		0.1	20	13.87	18.62
		0.5	0	23.58	9.28
		0.5	20	11.75	16.48
		0.9	0	21.05	6.73
		0.9	20	9.18	13.90
0.4	0.2	0.1	0	35.56	2.67
		0.1	30	16.68	12.35
		0.5	0	34.28	1.39
		0.5	30	14.14	9.80
		0.9	0	33.07	0.18
		0.9	30	10.83	6.49
0.4	0.4	0.1	0	41.01	8.13
		0.1	35	19.36	19.79
		0.5	0	38.31	5.42
		0.5	35	15.52	15.95
		0.9	0	34.45	1.55
		0.9	35	9.92	10.33

Table 3: Values for call and put semi-lookback spread options with terminal payouts of, respectively, $(\overline{S}_1 - S_2 - K)^+$ and $(K - \overline{S}_1 + S_2)^+$. We assume that $S_1(0) = S_2(0) = \overline{S}_1(0) = 100$, $r = 0.05\ \mathrm{yr}^{-1}$, $q_1 = q_2 = 0$, $t = 1$ yr.

$$\mathcal{P}(X_1(T) \in dx_1, X_2(T) \in dx_2, \underline{X}_1 \geq K_1, \overline{X}_2 \leq K_2)$$

$$\mathcal{P}(X_1(T) \in dx_1, X_2(T) \in dx_2, \underline{X}_1 \geq K_1, \underline{X}_2 \geq K_2)$$

Integrating these functions multiplied by the payoff function gives rise to the desired option premium.

Similar to the semi-lookbacks considered in the previous section, we can also evaluate a special class of barrier options which are European calls or puts written on one asset, subject to a condition that the value of another asset never hits a pre-determined boundary. The probability densities necessary for this type of options are

$$\mathcal{P}(X_1(T) \in dx_1, \overline{X}_2 \leq K_2)$$

$$\mathcal{P}(X_1(T) \in dx_1, \underline{X}_2 \geq K_2)$$

which can be obtained by integrating the relevant densities.[4]

4 Conclusions

We have presented a technique for pricing lookback options on two assets following correlated geometric Brownian motions. The essential part of this technique is to derive the probability density/distribution functions of the extreme values of two correlated Brownian motions. With this technique, the prices of many kinds of lookback and barrier options can be calculated efficiently. We hope that our pricing technology will be useful for any research in the future that involves the extreme values of two correlated geometric Brownian motions.

[4] Heynen and Kat (1994) have derived the same density function and obtained closed-form solutions for semi-barrier options.

Appendix:

PROOF OF THEOREM 1: Part (i) is a standard result and can be found in Rubinstein (1992). To prove (ii), the density function can be obtained by using a reflection principle argument. Karatzas and Shreve (1991) give the zero drift result, and our result just shifts by a Girsanov factor. Alternatively, one can obtain this result by solving the Fokker-Planck equation using a method of images procedure (see, for example, Wilmott, Dewynne, and Howison (1993)). To get the second density function, define $g(x)\,dx = \mathcal{P}\left(\underline{X}_1(t) > x_1, \overline{X}_1(t) < x_2, X_1(t) \in dx\right)$. Then $g(x)$ satisfies the following Fokker-Planck equation with absorbing boundaries:

$$\frac{\partial g}{\partial t} = \frac{1}{2}\sigma_1^2 \frac{\partial^2 g}{\partial x^2} - \alpha_1 \frac{\partial g}{\partial x}$$

$$g(x,0) = \delta(x), \quad g(x_1,t) = g(x_2,t) = 0$$

where $\delta(x)$ denotes the Dirac delta function with a spike at $x = 0$. A routine separation of variables technique leads directly to the answer given. See Gardiner (1990) for the zero-drift solution.

Integrating the expressions in part (ii) over $x \in [x_1, x_2]$ leads immediately to the joint distributions in (iii) for the minimum and maximum of a Brownian motion. ■

PROOF OF THEOREM 2: For notational convenience, we denote the density as $p(x_1, x_2, t)$. We know that p satisfies the Fokker-Planck equation

$$\frac{\partial p}{\partial t} = -\alpha_1 \frac{\partial p}{\partial x_1} - \alpha_2 \frac{\partial p}{\partial x_2} + \frac{1}{2}\sigma_1^2 \frac{\partial^2 p}{\partial x_1^2} + \rho\sigma_1\sigma_2 \frac{\partial^2 p}{\partial x_1 \partial x_2} + \frac{1}{2}\sigma_2^2 \frac{\partial^2 p}{\partial x_2^2}$$

with the following initial condition

$$p(x_1, x_2, t = 0) = \delta(x_1)\delta(x_2)$$

and absorbing boundary conditions

$$\begin{aligned} p(x_1 = m_1, x_2, t) &= 0 \\ p(x_1, x_2 = m_2, t) &= 0. \end{aligned}$$

We proceed to explicitly solve this PDE.

First, we note that we can eliminate the drift terms by the following transformation. Define

$$p(x_1, x_2, t) = e^{a_1 x_1 + a_2 x_2 + bt} q(x_1, x_2, t)$$

where a_1, a_2, and b are defined as above. Then $q(x_1, x_2, t)$ satisfies

$$\frac{\partial q}{\partial t} = \frac{1}{2}\sigma_1^2 \frac{\partial^2 q}{\partial x_1^2} + \rho\sigma_1\sigma_2 \frac{\partial^2 q}{\partial x_1 \partial x_2} + \frac{1}{2}\sigma_2^2 \frac{\partial^2 q}{\partial x_2^2}$$

with boundary conditions

$$\begin{aligned} q(x_1, x_2, t = 0) &= \delta(x_1)\delta(x_2) \\ q(x_1 = m_1, x_2, t) &= 0 \\ q(x_1, x_2 = m_2, t) &= 0. \end{aligned}$$

Next, we note that this PDE can be simplified by a suitable transformation of coordinates, to eliminate the cross-partial derivative and normalize the Brownian motions. Explicitly, if we define new coordinates z_1 and z_2, as given above, then

$$q(x_1, x_2, t) = \frac{h(z_1(x), z_2(x), t)}{\sigma_1\sigma_2\sqrt{1-\rho^2}}$$

and $h(z_1, z_2, t)$ satisfies

$$\frac{\partial h}{\partial t} = \frac{1}{2}\left(\frac{\partial^2 h}{\partial z_1^2} + \frac{\partial^2 h}{\partial z_2^2}\right)$$

with boundary conditions

$$\begin{aligned} h(z_1, z_2, t) &= \delta(z_1 - z_{10})\delta(z_2 - z_{20}) \\ h(L_1, t) &= h(L_2, t) = 0, \end{aligned}$$

where

$$\begin{aligned} L_1 &= \{(z_1, z_2) : z_2 = 0\} \\ L_2 &= \left\{(z_1, z_2) : z_2 = -\frac{\sqrt{1-\rho^2}}{\rho} z_1\right\} \end{aligned}$$

These boundary conditions along L_1 and L_2 are more conveniently expressed in polar coordinates. Introducing polar coordinates (r, θ) corresponding to (z_1, z_2) as defined above, we obtain

$$\frac{\partial h}{\partial t} = \frac{1}{2}\left(\frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r} + \frac{1}{r^2}\frac{\partial^2 h}{\partial \theta^2}\right)$$

with boundary conditions

$$\begin{aligned} h(r, \theta, t = 0) &= \frac{1}{r_0}\delta(r - r_0)\delta(\theta - \theta_0) \\ h(r, \theta = 0, t) &= 0 \\ h(r, \theta = \beta, t) &= 0. \end{aligned}$$

To solve this PDE for $h(r,\theta,t)$, we look for separable solutions of the form $R(r)\Theta(\theta)T(t)$. Plugging this in to the PDE, we find

$$\frac{T'}{T} = \frac{1}{2}\left(\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta}\right) \equiv -\lambda^2/2$$

where the separation constant is negative because the solutions must decay as $t \to \infty$. Hence, we have

$$T(t) \sim e^{-\lambda^2 t/2}$$

and

$$\left(r^2\frac{R''}{R} + r\frac{R'}{R} + \lambda^2 r^2\right) + \left(\frac{\Theta''}{\Theta}\right) = 0.$$

Defining $\Theta''/\Theta = -k^2$, we find

$$\Theta(\theta) \sim A\sin k\theta + B\cos k\theta.$$

The boundary conditions require that $\Theta(0) = \Theta(\beta) = 0$, and hence k must be real, $B = 0$, and

$$\sin k\beta = 0.$$

This last requirement restricts k to discrete values of the form

$$k_n = \frac{n\pi}{\beta}, \quad n = 1, 2, \ldots.$$

Thus the most general angular solution consistent with the boundary conditions is

$$\Theta(\theta) \sim \sin\frac{n\pi\theta}{\beta}, \quad n = 1, 2, \ldots.$$

Finally, the radial part of the solution is

$$r^2 R'' + rR' + (\lambda^2 r^2 - k_n^2)R = 0.$$

Defining $y = \lambda r$, we can rewrite this in the standard form

$$y^2\frac{d^2R}{dy^2} + y\frac{dR}{dy} + (y^2 - k_n^2)R = 0.$$

This is Bessel's equation, with the well-known fundamental solutions $J_{k_n}(y)$ and $I_{k_n}(y)$. Since $I_{k_n}(0)$ diverges, and we require $R(0)$ to be well-behaved, the $I_{k_n}(x)$ solution is not permitted. Hence the general radial solution is

$$R(r) \sim J_{k_n}(\lambda r).$$

In summary, then, the most general solution to the PDE for $h(r,\theta,t)$ consistent with the absorbing boundary conditions $h(r,0,t) = h(r,\beta,t) = 0$, is given by

$$h(r,\theta,t) = \int_0^\infty \sum_{n=1}^{\infty} c_n(\lambda) e^{-\frac{\lambda^2 t}{2}} \sin\left(\frac{n\pi\theta}{\beta}\right) J_{\frac{n\pi}{\beta}}(\lambda r) d\lambda.$$

Our goal now is to find the coefficients $c_n(\lambda)$ which fit the initial condition $h(r,\theta,0) = r_0^{-1}\delta(r - r_0)\delta(\theta-\theta_0)$.

To find $c_n(\lambda)$, multiply the previous equation at $t=0$ by $\sin\left(\frac{m\pi\theta}{\beta}\right)$ and integrate over θ. We find

$$r_0^{-1}\delta(r-r_0)\sin\left(\frac{m\pi\theta_0}{\beta}\right) = \frac{\beta}{2}\int_0^\infty d\lambda\, c_m(\lambda) J_{\frac{m\pi}{\beta}}(\lambda r).$$

Next, multiply this equation by $rJ_{\frac{m\pi}{\beta}}(\lambda' r)$ and integrate over r. Using the well-known completeness relation

$$\int_0^\infty xJ_\nu(ax)J_\nu(bx)\,dx = a^{-1}\delta(a-b),$$

we find

$$c_m(\lambda') = \frac{2\lambda'}{\beta}\sin\left(\frac{m\pi\theta_0}{\beta}\right) J_{\frac{m\pi}{\beta}}(\lambda' r_0).$$

Plugging this expression back into the general formula for h, we find

$$h(r,\theta,t) = \int_0^\infty \left(\frac{2\lambda}{\beta}\right)\sum_{n=1}^\infty e^{-\frac{\lambda^2 t}{2}}\sin\left(\frac{n\pi\theta_0}{\beta}\right)\sin\left(\frac{n\pi\theta}{\beta}\right) J_{\frac{n\pi}{\beta}}(\lambda r_0) J_{\frac{n\pi}{\beta}}(\lambda r) d\lambda.$$

The λ integral can be performed explicitly using the fact that [Gradshteyn and Ryzhik, p. 718]

$$\int_0^\infty xe^{-c^2x^2}J_\nu(ax)J_\nu(bx)\,dx = \frac{1}{2c^2}e^{-\frac{a^2+b^2}{4c^2}} I_\nu\left(\frac{ab}{2c^2}\right).$$

Doing so leads to the final expression that

$$h(r,\theta,t) = \left(\frac{2}{\beta t}\right)\sum_{n=1}^\infty e^{-\frac{r^2+r_0^2}{2t}}\sin\left(\frac{n\pi\theta_0}{\beta}\right)\sin\left(\frac{n\pi\theta}{\beta}\right) I_{\frac{n\pi}{\beta}}\left(\frac{rr_0}{t}\right).$$

This completes the proof of part (i). The proofs of parts (ii) and (iii) follow immediately by symmetry. ■

Proof of Corollary 1: Follow the proof of Theorem 2(i) until the PDE for $h(z_1,z_2,t)$ is derived. When $\rho_n = -\cos(\pi/n)$, note that the angles between the lines L_1 and L_2 in Theorem 2 take the special values

$$\beta_n = \pi/n, \quad n = 1,2,\ldots.$$

For these angles, a method of images solution to the PDE is possible, as follows. Note that

$$g^\pm(z_1,z_2,t;a_1,a_2) = \pm(2\pi t)^{-1}\exp\left(-\frac{1}{2}\left[(z_1-a_1)^2+(z_2-a_2)^2\right]\right)$$

satisfies the PDE with initial condition

$$g^\pm(z_1,z_2,0;a_1,a_2) = \pm\delta(z_1-a_1)\delta(z_2-a_2).$$

Furthermore, since the PDE is linear in h, any linear combination of these $g^{\pm}$'s, with different values of (a_1, a_2) also satisfies the PDE. We want to find that particular linear combination which also satisfies the boundary and initial conditions. Consider the case $n = 3$, as illustrated in Figure 4. For this correlation value, we have $\beta_3 = \pi/3$. Let a circle enclosing a '+' or '−' represent the solution $g^{\pm}$, with the location of center of the symbol indicating the value of (a_1, a_2). The first hextant, enclosed by solid radii (representing the rays L_1 and L_2), is the region $\theta \in [0, \pi/3]$, the region where we want to find a solution to the PDE. The '+' in the first hextant is located at (z_{10}, z_{20}), which makes an angle θ_0 with respect to the z_1-axis, and is at a distance r_0 from the origin. Since this is the only symbol in the first hextant, the delta function initial condition is satisfied. The other symbols are located as follows (each at distance r_0 from the origin): the '+' symbols occur at angles $\theta_0 + 2\pi/3$ and $\theta_0 + 4\pi/3$, and the '−' symbols occur at angles $-\theta_0$, $-\theta_0 + 2\pi/3$, and $-\theta_0 + 4\pi/3$. We claim that $h(z_1, z_2, t)$ is given by the sum of these six densities, each with unit weighting. As already seen, this linear combination satisfies the PDE and initial condition, and hence we only need to show that the absorbing boundary conditions are satisfied. But as is easily seen from the symmetry of the diagram, the six densities cancel in pairs along the rays L_1 and L_2. Hence the absorbing boundary conditions are satisfied, and the sum of these six Gaussians is the unique solution for $h(z_1, z_2, t)$. The solution for other values of n follows in a similar fashion, leading to the result given.

■

Proof of Theorem 3: Set

$$\begin{aligned} B_1(t) &= w_1(t) \\ B_2(t) &= -\frac{\rho}{\sqrt{1-\rho^2}} w_1(t) + \frac{1}{\sqrt{1-\rho^2}} w_2(t) \end{aligned}$$

Then, (B_1, B_2) is a standard (uncorrelated) two-dimensional Brownian motion.

(1) For density function u_{+0}, we have

$$\begin{aligned} &u_{+0}(x_1, x_2, t; \bullet)\, dx_1\, dx_2 \\ &\quad = \mathcal{P}(\max_{0\le s\le t}(\alpha_1 s + \sigma_1 w_1(s)) \le M_1, \alpha_1 t + \sigma_1 w_1(t) \in dx_1; \alpha_2 t + \sigma_2 w_2(t) \in dx_2) \\ &\quad = \mathcal{P}(\max_{0\le s\le t}(\alpha_1 s + \sigma_1 B_1(s)) \le M_1, \alpha_1 t + \sigma_1(t) B_1(t) \in dx_1; \alpha_2 t + \sigma_2(\rho B_1(t) + \sqrt{1-\rho^2} B_2(t)) \in dx_2) \\ &\quad = \mathcal{P}(\max_{0\le s\le t}(\alpha_1 s + \sigma_1 B_1(s)) \le M_1, \alpha_1 t + \sigma_1 B_1(t) \in dx_1; \sigma_2 B_2(t) \in \frac{dx_2 - \alpha_2 t - \rho\sigma_2 B_1(t)}{\sqrt{1-\rho^2}}) \\ &\quad = \mathcal{P}(\max_{0\le s\le t}(\alpha_1 s + \sigma_1 B_1(s)) \le M_1, \alpha_1 t + \sigma_1 B_1(t) \in dx_1; \sigma_2 B_2(t) \in \frac{dx_2 - \alpha_2 t - \rho\sigma_2(\frac{x_1 - \alpha_1 t}{\sigma_1})}{\sqrt{1-\rho^2}}) \end{aligned}$$

$$= \mathcal{P}(\max_{0\le s\le t}(\alpha_1 s + \sigma_1 B_1(s)) \le M_1, \alpha_1 t + \sigma_1 B_1(t) \in dx_1) \times \mathcal{P}(\sigma_2 B_2(t) \in \frac{dx_2 - \alpha_2 t - \rho\sigma_2(\frac{x_1 - \alpha_1 t}{\sigma_1})}{\sqrt{1-\rho^2}})$$

Equation (10) follows immediately from part 1 of Theorem 1. The proof for u_{-0} follows by symmetry.

(2) Similarly, for the density function f_{+0},

$$\begin{aligned}
& f_{+0}(x_1, x_2, t; \bullet)\, dx_1\, dx_2 \\
&\quad = \mathcal{P}(\max_{0\le s\le t}(\alpha_1 s + \sigma_1 w_1(s)) \in dx_1; \alpha_2 t + \sigma_2 w_2(t) \in dx_2) \\
&\quad = \mathcal{P}(\max_{0\le s\le t}(\alpha_1 s + \sigma_1 B_1(s)) \in dx_1; \alpha_2 t + \sigma_2(\rho B_1(t) + \sqrt{1-\rho^2} B_2(t)) \in dx_2) \\
&\quad = \mathcal{P}(\max_{0\le s\le t}(\frac{\alpha_1}{\sigma_1} s + B_1(s)) \in \frac{dx_1}{\sigma_1}; \frac{\alpha_2}{\sigma_2\rho} t + B_1(t) \in \frac{dx_2}{\sigma_2\rho} - \frac{\sqrt{1-\rho^2}}{\rho} B_2(t)) \\
&\quad = \int_{-\infty}^{\infty} \mathcal{P}(\max_{0\le s\le t}(\frac{\alpha_1}{\sigma_1} s + B_1(s)) \in \frac{dx_1}{\sigma_1}; \frac{\alpha_2}{\sigma_2\rho} t + B_1(t) \in \frac{dx_2}{\sigma_2\rho} - \frac{\sqrt{1-\rho^2}}{\rho} x | B_2(t) = x) \times \\
&\qquad\qquad \mathcal{P}(B_2(t) \in dx) \\
&\quad = \int_{-\infty}^{\infty} \mathcal{P}(\max_{0\le s\le t}(\frac{\alpha_1}{\sigma_1} s + B_1(s)) \in \frac{dx_1}{\sigma_1}; \frac{\alpha_1}{\sigma_1} t + B_1(t) \in \frac{dx_2}{\sigma_2\rho} - \frac{\sqrt{1-\rho^2}}{\rho} x + (\frac{\alpha_1}{\sigma_1} - \frac{\alpha_2}{\sigma_2\rho})t) \times \\
&\qquad\qquad \mathcal{P}(B_2(t) \in dx) \\
&\quad = \left[\int_a^{\infty} \frac{2(2\bar{y} - \bar{x})}{\sqrt{2\pi t^3}} \exp(-\frac{(2\bar{y}-\bar{x})^2}{2t}) \exp(\frac{\alpha_1}{\sigma_1}\bar{x} - \frac{1}{2}(\frac{\alpha_1}{\sigma_1})^2 t) \frac{1}{\sqrt{2\pi t}} \exp(-\frac{x^2}{2t}) dx\right] \frac{dx_1\, dx_2}{\rho\sigma_1\sigma_2}
\end{aligned}$$

where we have set $\bar{y} = \frac{x_1}{\sigma_1}$ and $\bar{x} = \frac{x_2}{\sigma_2\rho} - \frac{\sqrt{1-\rho^2}}{\rho} x + (\frac{\alpha_1}{\sigma_1} - \frac{\alpha_2}{\sigma_2\rho})t)$. In the last equation, we used the density corresponding to the distribution given in Theorem 1. The lower integral bound a is defined by the condition $\bar{y} \ge \bar{x}$ which is equivalent to

$$x \ge \frac{\rho}{\sqrt{1-\rho^2}}(\frac{x_2}{\sigma_2\rho} - \frac{x_1}{\sigma_1} + (\frac{\alpha_1}{\sigma_1} - \frac{\alpha_2}{\sigma_2\rho})t) \equiv a$$

Evaluating the integral yields the result given. The result for $f_{0-}(x_1, x_2, t)$ follows by symmetry. ■

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