

A Variable Reduction Technique for Pricing Average-Rate Options*

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1 Introduction

“Average-rate options”, commonly known as Asian options, are contingent claims whose payoffs depend on the arithmetic average of some underlying index (e.g., stock prices, exchange rates or interest rates) over a fixed time horizon. While no average-rate options are traded as standardized option contracts in any organized options or futures exchange in the world, these options, especially those with the underlying being exchange rates or interest rates, are extremely popular in the over-the-counter market among institutional investors.

There are many economic reasons why average-rate options are so popular. For example, if a corporation expects to receive or pay foreign currency claims on a regular basis, then a foreign currency option based on an average of exchange rates represents one way to reduce its average foreign currency exposure. A similar argument can be made for an interest rate option based on an average of short term interest rates (e.g., 3 month LIBOR) or an average of longer term interest rates (e.g., 10 swap rates). Since the average of the underlying tends to be much less volatile than the underlying itself, average-rate options are priced more cheaply than the standard (plain vanilla) options. This reduces significantly the hedging costs for corporations in need of average-rate options. In addition, by its very design, the payoff of an average-rate option is less dependent on the closing prices of the underlying near the expiration date. Thus, it reduces the significance of market impact or price manipulation at the maturity of the option.

The pricing and hedging of average-rate options raise some interesting issues. First, these options are path-dependent, i.e., the value of an average-rate option at any point in time depends not only upon the value of the underlying at that time but also upon the history of the underlying up to that time. Specifically, if the underlying follows a Markov-diffusion process, then the no-arbitrage value of an average-rate option would depend upon the current level of the underlying as well as the average levels of the underlying up to that time. Thus, when applying standard option pricing techniques (such as the binomial method or the finite difference method for solving a partial differential equation (PDE)), a second state variable (in addition to the underlying itself) is often necessary. This makes the pricing problem much more complicated. Second, the arithmetic average is not lognormally distributed when the underlying follows a standard lognormal process. In fact, it is impossible to uncover analytically the probability distribution of the arithmetic average when the underlying is lognormally distributed. Due to the above reasons, it is well known that no analytical solution exists for the price of European calls or puts written on the arithmetic average when

the underlying index follows a lognormal process.¹ Consequently, numerical techniques must be relied upon in order to determine the no-arbitrage value of average-rate options.

There are several types of numerical techniques that have become popular for valuing average-rate options. The first one, which perhaps is also the most simple and commonly used one, is the Monte Carlo simulations method originally discussed in Kemma and Vorst (1990) for the case when the underlying is a lognormal process. The Monte Carlo simulations method is convenient and flexible. In particular, it is applicable as long as the underlying follows a Markov-diffusion process. For example, it can be applied to a square root process for interest rates as well. However, in terms of the computing time required, this method is not very efficient.

The second class of numerical techniques for average-rate options employs the idea that an arithmetic average can be reasonably approximated by a geometric average with an appropriately adjusted mean and variance. This technique includes i) the modified-strike method (Vorst, 1990), which replaces the arithmetic average by a geometric average with an adjustment in the strike price to correct the mean bias; ii) the modified-geometric method (Kunitomo and Takahashi, 1992), which replaces the arithmetic average by a geometric average with its mean and variance adjusted to match the mean and variance of the arithmetic average; iii) the geometric conditioning method, which replaces the arithmetic average by its conditional expectation conditioning on the geometric average (Curran, 1992); and iv) the Edgeworth series expansion method (Turnbull and Wakeman, 1991), which applies an expansion of the distribution of the arithmetic average around the distribution of the geometric average (which is lognormal). All of the above mentioned methods have been shown to be reliable whenever the volatility of the underlying is not too large (e.g., less than 30%). However, numerical errors can become significant when the volatility is high.

The third class of numerical techniques for average-rate options attempts to address the arithmetic average in a more direct way. Specifically, these techniques put forward various discrete time models (e.g., binomial trees or finite difference grids) to approximate the continuous time value of the average-rate options. Hull and White (1993) have developed an extended Binomial method in which they construct a binomial tree with a vector of average rates stacking at each node. Conditioning on the current value of the average rate, they apply the standard recursive valuation method for each level of average rates chosen in the vector of average rates. A similar idea has been carried out by Dewynne and Wilmott (1993) in solving

¹Geman and Yor (1993) have developed a semi-analytical valuation method using the Laplace transformation technique.

numerically the partial differential equation for average-rate options. Carverhill and Clewlon (1990) have developed another approach using the Fourier transformation technique. Their approach involves calculating the distribution function of the arithmetic average through the Fast Fourier Transform technique. All of the three methods mentioned above require intensive computing time, as they have to handle more or less a two-dimensional valuation problem.

In this paper we propose a new valuation technique, called the variable reduction technique, for average rate options. This method has many advantages over the various techniques described above. The main idea of our method is quite simple. Basically, this method transforms the valuation problem of an average-rate option into an evaluation of a conditional expectation that is determined by a one-dimensional Markov process (as suppose to a two-dimensional Markov process commonly known).² This transformation is extremely useful since numerically it is much easier to handle a one-dimensional valuation problem than a two-dimensional problem. Alternatively, we can also derive a partial differential equation that the value function of an average-rate option must be satisfied. In this case, the PDE is a second order parabolic one with one state variable and one time variable. Standard numerical techniques can be applied to evaluate the conditional expectation or to solve numerically the partial differential equation that determines the value of an average-rate option.

Compared to the geometric approximation technique, our variable reduction technique works directly with the arithmetic average, and therefore will not encounter the cited approximation errors when the volatility of the underlying is relatively large. Furthermore, compared to the methods proposed by Hull and White (1993) and Dewynne and Wilmott (1993), our technique has reduced the dimensionality of pricing average-rate options by one, which certainly will make our pricing more efficient in terms of computing time. Finally, there is no doubt that this technique is more favorable than the Monte Carlo method when the underlying is a lognormal process. Unfortunately, when the underlying is not lognormal, the variable reduction technique is no longer applicable for average-rate options.

The rest of the paper is organized as follows. In the next section, we illustrate the variable reduction technique in the simple Black-Scholes' economy in which there is one risky

²Ingersoll (1987) has shown that an average-strike option, i.e., an option whose strike price is the average of the underlying over a fixed horizon, can be handled by factoring out the average-strike and thereby reducing the two-state variables problem into a one-state variable problem. Wilmott, Dewynne and Howison (1993) have provided a similar variable reduction technique to the partial differential equation that the value of an average-strike option must be satisfied. However, Wilmott, Dewynne and Howison (1993) claim that the same technique doesn't work for average-rate options.

asset and one riskless bond. While much of the analysis in the paper assumes continuous averaging or continuous fixing, we will briefly discuss in this section the implementation of discrete averaging. In Section 3, we apply the variable reduction technique to average-rate options where the underlying index is an interest rate (for example, LIBOR rates with a constant maturity). Numerical comparisons of different methods are presented in Section 4. We conclude the paper in Section 5.

2 Average-Rate Options in the Black-Scholes Economy

To illustrate the idea of the variable reduction technique, we first consider the Black and Scholes' economy (Black and Scholes, 1973) in which there is one riskless bond and one risky asset, and the prices of the riskless bond and risky asset are determined as follows,

$$\begin{aligned} B_t &= e^{rt} \\ S_t &= S_0 e^{(\mu - q - \frac{1}{2}\sigma^2)t + \sigma w_t}, \quad t \geq 0 \end{aligned}$$

and w is a standard Brownian motion defined on a probability space, coefficients r , μ , q and σ are constants, and q is the implicit payout rate. The risky asset here could be a stock, foreign currency or commodity. The payout rate therefore would be the dividend yield, the foreign riskless rate, and the convenience yield, respectively. We assume that there exists a risk neutral probability or equivalent martingale measure Q under which the price of the risky asset is determined by

$$S_t = S_0 e^{(r - q - \frac{1}{2}\sigma^2)t + \sigma \tilde{w}_t}$$

where \tilde{w} is a standard Brownian motion under Q .

An average-rate European call option is defined to be an option that gives the holder the right (but not the obligation) to receive at the expiration date the arithmetic average of the price of the underlying asset over a finite time horizon for a fixed strike price. A similar definition can be made for an average-rate European put option. In this paper we will not consider average-strike options whose strike price is a fixed percentage of the average of the price of the risky asset over a fixed time horizon. However, the variable reduction technique works for average-strike options as well as shown in Ingersoll (1987) and Wilmott, Dewynne and Howison (1993). However, none of these two works were able to extend their approaches to average-rate options. In fact, Wilmott, Dewynne and Howison claimed that their variable reduction technique doesn't work for average-rate options.

Following Cox and Ross (1975) and Harrison and Kreps (1979), the price at time t of an average-rate call option, $C(t)$, with a maturity date T and a strike price K can be evaluated by

$$C(t) = e^{-r(T-t)} \mathbf{E}_t^* \left[\left(\frac{1}{T} A_T - K \right)^+ \right] \quad (1)$$

where \mathbf{E}_t^* denotes the conditional expectation under the risk neutral probability distribution, or equivalently, the equivalent martingale measure Q , conditional on the information set at time t , and

$$A_t = \int_0^t S_u du$$

A similar definition can be made for an average-rate put option.

We note that

$$\begin{aligned} dS_t &= rS_t dt + \sigma S_t d\tilde{w}_t \\ dA_t &= S_t dt \end{aligned}$$

under the risk neutral probability distribution Q , and therefore S and A together form a two dimensional Markov process under Q . Thus, the value of an average-rate call option at t ($< T$) must be a function of S_t , A_t and t , i.e., $C = C(S, A, t)$. Moreover, since $e^{-rt} C$ must be a martingale under Q , the drift of $e^{-rt} C$ under Q must be zero. This leads to the partial differential equation for C ,

$$\frac{\sigma^2}{2} S^2 C_{SS} + (r - q) S C_S + S C_A + C_t - rC = 0 \quad (2)$$

This is a second-order partial differential equation (PDE) with two space variables and one time variable. Moreover, the second order partial derivative with respect to A is degenerate. Numerical solutions of this partial differential equation is possible but cumbersome as well as time-consuming.

We now introduce the variable reduction method which transforms (2) into a PDE with only one state variable and one time variable. To motivate our transformation, let us re-write the valuation equation (1) as follows,

$$\begin{aligned} C(t) &= e^{-r(T-t)} \mathbf{E}_t^* \left[\frac{1}{T} A_t - K + \frac{1}{T} \int_t^T S_u du \right]^+ \\ &= \frac{S_t}{T} e^{-r(T-t)} \mathbf{E}_t^* \left[x_t + \int_t^T \frac{S_u}{S_t} du \right]^+ \end{aligned}$$

where we have introduced a new state variable x determined by

$$x_t = \frac{1}{S} (A_t - TK)$$

Since S_u/S_t ($u > t$) is independent of the history of S up to t , the conditional expectation in the above equation must be a function of x_t . Thus, C can be written as a function of x_t and t multiplied by S_t , i.e.,

$$C(S_t, A_t, t) = S_t f(x_t, t)$$

for some function f of x and t only:

$$f(x, t) = \frac{e^{-r(T-t)}}{T} \mathbf{E}_t^* \left[x + \int_t^T \frac{S_u}{S_t} du \right]^+$$

A simple calculation shows

$$\begin{aligned} C_S &= f - x f_x \\ C_{SS} &= \frac{1}{S} x^2 f_{xx} \\ C_A &= f_x \\ C_t &= S f_t \end{aligned}$$

Substituting these relations into the above PDE for C , we get

$$S \times \left[\frac{\sigma^2}{2} x^2 f_{xx} + (1 - \alpha x) f_x + f_t - qf \right] = 0$$

where $\alpha = r - q$. Since $S_t > 0$ for all t , we obtain the partial differential equation for f ,

$$\frac{1}{2} \sigma^2 x^2 f_{xx} + (1 - \alpha x) f_x + f_t - qf = 0 \quad (3)$$

The boundary condition is given by

$$f(x, T) = \frac{1}{T} \max[x, 0] \quad (4)$$

The value of the call option at time t is then given by $S_t f(x_t, t)$. In other words, the variable reduction technique has helped us to factor out S from the call price. We summarize our results in the following proposition.

Proposition 1 *The value of an average-rate call option is determined by $S_t f(x_t, t)$, where f satisfies the PDE (3) and the boundary condition (4), and where $x_t = (A_t - TK)/S_t$.*

It is useful to note that the stochastic process x is a diffusion process by itself, i.e.,

$$dx_t = (1 - \alpha x_t + \sigma^2 x_t) dt - \sigma x_t d\tilde{w}_t$$

This explains why we are able to factor out S from the PDE (2) to get the PDE (3). Moreover, if we introduce a pseudo probability measure Q' in such a way that

$$dx_t = (1 - \alpha x_t)dt - \sigma x_t dw'_t$$

where w' is a standard Brownian motion under Q' , then (3) is equivalent to the statement that under Q' , the discounted value (f) is a martingale, while the discount rate is the implicit payout rate q , i.e.,

$$f(x_t, t) = \mathbf{E}'_t \left[\frac{e^{-q(T-t)}}{T} \max[x_T, 0] \right] \quad (5)$$

where the expectation is taken under Q' . The above formula is also called the Feynman-Kac representation of the partial differential equation (3), see Karatzas and Shreve (1988).

For readers who are familiar with the Harrison and Kreps' argument, it can be shown that $e^{rt}/S_t e^{qt}$ is a martingale under Q' , i.e., the riskless asset price discounted by the risky asset price (after adjusted by the payout rate) is a martingale under Q' . Thus, Q' is the equivalent martingale measure when the risky asset is chosen as a numeraire. Moreover, the martingale argument allow us to claim that

$$\frac{C_t}{S_t e^{qt}} = \mathbf{E}'_t \left[\frac{\frac{A_T}{T} - K}{S_T e^{qT}} \right]^+$$

Introducing x_t as we did above and realizing that x is a Markov process by itself under Q' , we can immediately conclude that the right-hand side of the above equation must be a function of x_t and t . In other words, S can be factored out using the martingale argument as well.

We also note that when $x_t \geq 0$ ($t < T$), one can easily obtain an explicit formula for f ,

$$f(x, t) = \frac{1}{T} e^{-r(T-t)} x + \frac{e^{-q(T-t)} - e^{-r(T-t)}}{T(r - q)} \quad (6)$$

However, for $x_t < 0$, we must utilize standard numerical techniques such as the finite difference method or the Monte carlo simulations method to evaluate f from the PDE (3) or from the conditional expectation that defines f in (5). Note that if we are to apply the finite difference method, then we need to use the boundary conditions (6), $\lim_{x \rightarrow -\infty} f(x, t) = 0$ for small values of x and (4) for large value of x .

Our variable reduction technique can also be applied to average-rate options where the averaging is taken at a discrete set of time points, i.e., discrete averaging or discrete fixing. For illustrations, we assume that averaging takes place at points $0 = t_1 < t_2 < t \dots < t_n = T$.

For simplicity, let us consider evaluating the option exactly on the points where averaging takes place.³ Define

$$\begin{aligned} A_{t_k} &\equiv \sum_{i=1}^k S_{t_i}, \\ x_{t_k} &\equiv \frac{A_{t_k} - nK}{S_{t_k}}, \quad k = 1, 2, \dots, n \end{aligned}$$

Applying the same variable reduction technique, we can show that

$$\begin{aligned} C_{t_k} &= e^{-r(t_n-t_k)} \frac{S_{t_k}}{n} \mathbf{E}_{t_k}^* \left[x_{t_k} + \sum_{i=k+1}^n \frac{S_{t_i}}{S_{t_k}} \right]^+ \\ &\equiv S_{t_k} f(x_{t_k}, t_k) \end{aligned}$$

where

$$\begin{aligned} f(x_{t_k}, t_k) &= \frac{e^{-r(t_n-t_k)}}{n} \mathbf{E}_{t_k}^* \left[x_{t_k} + \sum_{i=k+1}^n \frac{S_{t_i}}{S_{t_k}} \right]^+, \quad k = 1, 2, \dots, n-1 \\ f(x_{t_n}, t_n) &= \frac{1}{n} x_{t_n}^+ \end{aligned}$$

It then follows that

$$\begin{aligned} f(x_{t_k}, t_k) &= e^{-r(t_{k+1}-t_k)} \mathbf{E}_{t_k}^* \left[\frac{S_{t_{k+1}}}{S_{t_k}} f(x_{t_{k+1}}, t_{k+1}) \right] \\ x_{t_{k+1}} &= \frac{S_{t_k}}{S_{t_{k+1}}} x_{t_k} + 1 \end{aligned} \tag{7}$$

We note that

$$\begin{aligned} \frac{S_{t_{k+1}}}{S_{t_k}} &= e^{(r-q-\frac{\sigma^2}{2})\Delta t_k + \sigma\sqrt{\Delta}\epsilon} \\ \frac{S_{t_k}}{S_{t_{k+1}}} &= e^{-(r-q-\frac{\sigma^2}{2})\Delta t_k - \sigma\sqrt{\Delta}\epsilon} \end{aligned}$$

where ϵ is a random variable distributed as $N(0, 1)$, and $\Delta t_k = t_{k+1} - t_k$. Thus, (7) can be solved recursively by numerical integrations.⁴

Alternatively, we can also determine f by

$$f(x_{t_k}, t_k) = \frac{e^{-q(t_n-t_k)}}{n} \mathbf{E}'_t [\max[x_{t_n}, 0]]$$

³A similar approach can be used to value the option at times other than those averaging points.

⁴Specifically, we can fix a set of grid points for x , and evaluate f over these points recursively. For those points that are not on the grids, a second order interpolation can be used to find the value of f on these points.

This formula is useful if we would like to value f by Monte-Carlo simulation. Note that in this case we will be simulating the process x under the probability measure Q' .

It is important to note that as in the case of continuous averaging, the valuation problem here is also a one-dimensional problem. We have avoided the complexity of the two dimensional problem encountered by Hull and White (1993) and Dewynne and Wilmott (1993).

Before leaving this section, we point out that all of our analyses so far are equally applicable when σ and r are functions of t . This suggests that we can value average-rate options when we have a deterministic term structure of volatilities and interest rates.

3 Average-Rate Options on Interest Rates

We now apply the variable reduction technique to value average-rate options on interest rates related derivative instruments. Such instruments are commonly traded in the over-the-counter (OTC) markets, and have played an important role in satisfying various needs of institutional investors or borrowers. Among those interest rates related derivative instruments, options on the average of CMT or CMS rates (constant maturity treasury yields or constant maturity swap rates) have been somewhat popular. Those option contracts can also be imbedded in a swap transaction to serve as speculative or hedging purposes for the investors or the issuers.

In this section we present the variable reduction technique for pricing the average-rate options on CMS rates. First, let us define an option on the average of CMS rates (with a fixed time to maturity). Let $L^\tau(t)$ denote the yield at time t for a zero coupon bond with a time to maturity of τ years.⁵ Then, the average of $L^\tau(t)$ in a prespecified time period $[0, T]$ is given by

$$Z(T) = \frac{1}{T} \int_0^T L^\tau(t) dt$$

where τ is a fixed real number, e.g., 0.25, 0.5, or 1. Let $P(t, T)$ denote the price at time t of a zero coupon bond maturing at time T . Then, we can re-express $L^\tau(t)$ as

$$L^\tau(t) = \left(\frac{1}{P(t, t + \tau)} - 1 \right) \frac{1}{\tau}$$

The payoff of a European call option on the average rates at the expiration date T with a strike price K is given by

$$C(T) = \max[Z(T) - K, 0]$$

⁵For simplicity, we will not consider pricing average rate options based on par yields.

while the payoff for a European put option is given by

$$C(T) = \max[K - Z(T), 0]$$

In the rest of this section, we determine the arbitrage-free value of such European call or put options by using the similar technique introduced in the previous section. Note that once the call price is obtained, the value of a put option can be easily derived through the “put-call parity”, which will be shown later in this section.

3.1 Arbitrage-Free Forward Rate Processes

To evaluate an average-rate option on interest rates with a constant maturity, we employ the Heath-Jarrow-Morton’s model as our basic model for term structure of interest rates. This model is based on an explicit specification of the instantaneous forward rates and a restriction of no arbitrage, see Heath, Jarrow, and Morton (1992) for details. In this setting, the instantaneous forward rate process under the equivalent martingale measure is described as

$$f(t, T) = f(0, T) + \sum_{i=1}^N \int_0^t \left(\sigma_i(s, T) \int_s^T \sigma_i(s, u) du \right) ds + \sum_{i=1}^N \int_0^t \sigma_i(s, T) d\tilde{w}_i(s)$$

where w is an N -dimensional standard Brownian motions. In the above specification, the diffusion term or the volatility process σ can be chosen by the user (subject to a regularity condition) while the drift is completely determined by the choice of σ , due to the no arbitrage condition. In particular, the spot rate process, $r(t) = f(t, t)$, is given by

$$r(t) = f(0, t) + \sum_{i=1}^N \int_0^t \left(\sigma_i(s, t) \int_s^t \sigma_i(s, u) du \right) ds + \sum_{i=1}^N \int_0^t \sigma_i(s, t) d\tilde{w}_i(s)$$

Given the spot rate, any interest rate contingent claim can be priced through the well-known property that the value process relative to the money market account is a martingale under the equivalent martingale measure or the risk neutral probability:

$$\frac{V(t)}{B(t)} = \mathbf{E}_t \left[\frac{V(T)}{B(T)} \right]$$

where $B(t) = \exp[\int_0^t r(s) ds]$, and $B(t)$ denotes the value process of the money market account.

3.2 A Constant Volatility Model

For simplicity, we shall specify a one-factor model of forward rates (with a constant volatility) in order to evaluate the average-rate options under consideration. That is, we set $N = 1$ in the forward rate process described above. The volatility function in the forward rate process is given by $\sigma_1(s, t) = \sigma$, where σ is a positive constant. This model is known to be a continuous time version of the Ho and Lee (1986)'s model. Specifically, the forward rate process can be described as

$$f(t, T) = f(0, T) + \sigma^2(Tt - \frac{t^2}{2}) + \sigma\tilde{w}(t)$$

and the spot rate process is given by

$$r(t) = f(0, t) + \frac{\sigma^2 t^2}{2} + \sigma\tilde{w}(t)$$

A straightforward calculation shows that the price at time t of a zero coupon bond maturing at time T is given by

$$P(t, T) = \frac{P(0, T)}{P(0, t)} \exp \left[-\frac{\sigma^2}{2} Tt(T - t) - \sigma(T - t)\tilde{w}(t) \right] \quad (8)$$

3.3 Pricing Average-Rate Options on Interest Rates

We will evaluate the average-rate options on interest rates under a term structure model with a constant volatility. As in the average rate options under the Black-Scholes economy, we will show that the valuation problem can be simplified to a partial differential equation with a single state variable and a time variables, after a simple transformation of variables. It is much easier to solve this equation numerically, for example, by the finite difference method. Moreover, in the special case where the option is deep-in-the money, an explicit valuation formula is obtained. As mentioned before, we will consider only the call option case, as the put option can be priced through a "put-call parity".

First, we will rewrite our valuation problem in terms of zero coupon bonds. Then, the final payoff of the call option is re-expressed as

$$\begin{aligned} C(T) &= \max[Z(T) - K, 0] \\ &= \max\left[\frac{1}{T} \int_0^T \left\{ \frac{1}{P(t, t + \tau)} - 1 \right\} \frac{1}{\tau} dt - K, 0 \right] \\ &= \frac{1}{T\tau} \max\left[\int_0^T \frac{1}{P(t, t + \tau)} dt - k, 0 \right] \end{aligned}$$

where τ is a positive constant and $k = (1 + K\tau)T$.

By using the expression (8), the reciprocal of the price of a zero coupon bond with τ years to maturity, $\frac{1}{P(t, t+\tau)}$, is described as

$$\frac{1}{P(t, t+\tau)} = \frac{P(0, t)}{P(0, t+\tau)} \exp \left[\frac{\sigma^2}{2} (t+\tau)t\tau + \sigma\tau\tilde{w}(t) \right]$$

Hence, the price of the average-rate call option at time t (before the maturity date T) is given by

$$\begin{aligned} C(t) &= \mathbf{E}_t^* \left[\exp \left(- \int_t^T r(u)du \right) C(T) \right] \\ &= \frac{1}{T\tau} \frac{P(0, T)}{P(0, t)} e^{-\frac{\sigma^2(T^3-t^3)}{6}} \mathbf{E}_t^* \left[e^{-\sigma \int_t^T \tilde{w}(u)du} \times C(T) \right] \end{aligned} \quad (9)$$

where we use the relation, $\exp(-\int_t^T f(0, u)du) = P(0, T)/P(0, t)$, and where

$$C(T) = \max \left[\int_0^T \frac{P(0, u)}{P(0, u+\tau)} \exp \left[\frac{\sigma^2}{2} (u+\tau)u\tau + \sigma\tau\tilde{w}(u) \right] du - k, 0 \right]$$

Our main objective is to evaluate the conditional expectation in (9). Although it is a fairly tough task to evaluate this expectation directly, if the new variable defined below is introduced, the problem becomes much easier to handle. We will give this transformation in the next lemma.

Lemma 1 *Define a stochastic process X as*

$$X(t) = \frac{\int_0^t \frac{P(0, u)}{P(0, u+\tau)} e^{\frac{\sigma^2}{2}(u+\tau)u\tau + \sigma\tau\tilde{w}(u)} du - k}{e^{\sigma\tau\tilde{w}(t)}}$$

Then, X satisfies the following stochastic differential equation,

$$dX_t = \left(\frac{P(0, t)}{P(0, t+\tau)} e^{\frac{\sigma^2}{2}(t+\tau)t\tau} + \frac{(\sigma\tau)^2}{2} X_t \right) dt - \sigma\tau X_t d\tilde{w}(t) \quad (10)$$

where $X(0) = -k$, and the value at time t of the average-rate call can be expressed as

$$\begin{aligned} C(t) &= \frac{1}{T\tau} e^{-\int_t^T f(0, u)du} e^{-\frac{\sigma^2(T^3-t^3)}{6}} e^{\sigma(t-T+\tau)\tilde{w}(t)} \times \\ &\quad \mathbf{E}_t^* \left[e^{-\sigma \int_t^T (\tilde{w}(u) - \tilde{w}(t))du} \times \right. \\ &\quad \left. \max \left[X_t + \int_t^T \frac{P(0, u)}{P(0, u+\tau)} e^{\frac{\sigma^2}{2}(u+\tau)u\tau + \sigma\tau[\tilde{w}(u) - \tilde{w}(t)]} du, 0 \right] \right] \end{aligned} \quad (11)$$

The expectation on the right-hand side is a function of X_t .

PROOF. Equation (10) can be shown by Ito's lemma while (11) can be verified easily. The last statement follows from the fact that $\tilde{w}(u) - \tilde{w}(t)$ is independent of $w(t)$. ■

Here, we note that $\tilde{w}(t)$ can be expressed in terms of r_t . Therefore, $C(t)$ can be separated into the product of a function of (r, t) and a function of (X, t) .

$$C(r, X, t) = g(r, t)h(X, t) \quad (12)$$

where

$$\begin{aligned} g(r, t) &= \frac{1}{T\tau} \frac{P(0, T)}{P(0, t)} e^{-\frac{\sigma^2(T^3-t^3)}{6}} \times e^{(t-T+\tau)[r_t - f(0, t) - \frac{(\sigma t)^2}{2}]} \\ h(X, t) &= \mathbf{E}_t^* \left[e^{-\sigma \int_t^T (\tilde{w}(u) - \tilde{w}(t)) du} \times \max[X_t + P_u^{u+\tau} \int_t^T e^{\frac{\sigma^2}{2}(u+\tau)u\tau + \sigma\tau[\tilde{w}(u) - \tilde{w}(t)]} du, 0] \right]. \end{aligned}$$

where $P_u^{u+\tau} = \frac{P(0, u)}{P(0, u+\tau)}$. Clearly, $g(r, t)$ is calculated using the information available at time t while $h(X, t)$ turns out to satisfy a partial differential equation as shown in the following proposition.

Proposition 2 *h satisfies the partial differential equation*

$$\begin{aligned} \frac{(\sigma\tau)^2}{2} X^2 h_{XX} + \left[\frac{P(0, t)}{P(0, t+\tau)} e^{\frac{\sigma^2}{2}(t+\tau)t\tau} + \{\sigma^2\tau(T-t) - \frac{(\sigma\tau)^2}{2}\} X_t \right] h_X + \\ h_t + \frac{\sigma^2(t+\tau-T)^2}{2} h = 0 \end{aligned} \quad (13)$$

with the boundary condition

$$h(X, T) = \max[X_T, 0]. \quad (14)$$

PROOF. Since $C(t) \exp(-\int_0^t r(s) ds)$ is a martingale under the equivalent martingale measure, the drift of $C(t) \exp(-\int_0^t r(s) ds)$ must be 0. Recall that

$$\begin{aligned} dr &= \{f_t(0, t) + \sigma^2 t\} dt + \sigma d\tilde{w}(t) \\ dX &= \left(\frac{P(0, t)}{P(0, t+\tau)} e^{\frac{\sigma^2}{2}(t+\tau)t\tau} + \frac{(\sigma\tau)^2}{2} X \right) dt - \sigma\tau X d\tilde{w}(t). \end{aligned}$$

It is easily seen by the Itô's lemma that

$$\begin{aligned} \frac{\sigma^2}{2} C_{rr} + \frac{(\sigma\tau X)^2}{2} C_{XX} - \sigma^2\tau X C_{rX} + [f_t(0, t) + \sigma^2 t] C_r \\ + \left[\frac{P(0, t)}{P(0, t+\tau)} e^{\frac{\sigma^2}{2}(t+\tau)t\tau} + \frac{(\sigma\tau)^2}{2} X_t \right] C_X + C_t - rC = 0 \end{aligned} \quad (15)$$

Since $C(r, X, t) = g(r, t)h(X, t)$, simple calculations show

$$\begin{aligned}
C_r &= g_r h = (t + \tau - T)gh \\
C_{rr} &= g_{rr} h = (t + \tau - T)^2 gh \\
C_{rX} &= g_r h_X = (t + \tau - T)gh_X \\
C_X &= gh_X \\
C_{XX} &= gh_{XX} \\
C_t &= g_t h + gh_t = r gh - (t + \tau - T)[f_t(0, t) + \sigma^2 t] gh + gh_t.
\end{aligned}$$

Substituting the above relations into (15), we obtain

$$\begin{aligned}
g \times \left[\frac{(\sigma\tau X)^2}{2} h_{XX} + \left(\frac{P(0, t)}{P(0, t + \tau)} e^{\frac{\sigma^2}{2}(t+\tau)t\tau} + \{ \sigma^2\tau(T - t) - \frac{(\sigma\tau)^2}{2} \} X_t \right) h_X \right. \\
\left. + h_t + \frac{\sigma^2(t + \tau - T)^2}{2} h \right] = 0. \tag{16}
\end{aligned}$$

Next, noting that $g(r, t) > 0$ for all t and r , we obtain the desired partial differential equation for h . Finally, it is easily seen that the terminal boundary condition is given by

$$h(X, T) = \max[X_T, 0]$$

■

When $X_t \geq 0$, i.e., when the option is very deep-in-the money, we can show that h can be calculated explicitly. We present this result in the following proposition.

Proposition 3 *When $X_t \geq 0$, the price of an average-rate call option on interest rates is given by*

$$C(r, x, t) = g(r, t)h(X, t)$$

where $g(r, t)$ is defined as above and $h(X, t)$ is given by

$$\begin{aligned}
h(X, t) &= X_t e^{\frac{\sigma^2(T-t)^3}{6}} + \exp \left(\frac{\sigma^2}{2}(t + \tau)t\tau + \frac{\sigma^2(T - t)^3}{6} \right) \\
&\times \int_0^{T-t} \frac{P(0, s + t)}{P(0, s + t + \tau)} \exp \left[\sigma^2\tau s^2 + \sigma^2(\tau^2 + 2t\tau - T\tau)s \right] ds. \tag{17}
\end{aligned}$$

PROOF. Note that when $X_t \geq 0$,

$$\begin{aligned}
&\max \left[X_t + \int_t^T \frac{P(0, u)}{P(0, u + \tau)} e^{\frac{\sigma^2}{2}(u+\tau)u\tau + \sigma\tau[\bar{w}(u) - \bar{w}(t)]} du, 0 \right] \\
&= X_t + \int_t^T \frac{P(0, u)}{P(0, u + \tau)} e^{\frac{\sigma^2}{2}(u+\tau)u\tau + \sigma\tau[\bar{w}(u) - \bar{w}(t)]} du.
\end{aligned}$$

Hence

$$h(X, t) = X_t \mathbf{E}_t^* \left[e^{-\sigma \int_t^T [\tilde{w}(u) - \tilde{w}(t)] du} \right] + \mathbf{E}_t^* \left[e^{-\sigma \int_t^T [\tilde{w}(u) - \tilde{w}(t)] du} \int_t^T \frac{P(0, u)}{P(0, u + \tau)} e^{\frac{\sigma^2}{2}(u+\tau)u\tau + \sigma\tau[\tilde{w}(u) - \tilde{w}(t)]} du \right].$$

By the strong Markov property of the Brownian motion,

$$h(X, t) = X_t \mathbf{E}^* \left[e^{-\sigma \int_0^{T-t} \tilde{w}(s) ds} \right] + \mathbf{E}^* \left[e^{-\sigma \int_0^{T-t} \tilde{w}(s) ds} \int_0^{T-t} \frac{P(0, s+t)}{P(0, s+t+\tau)} e^{\frac{\sigma^2}{2}(s+t+\tau)(s+t)\tau + \sigma\tau\tilde{w}(s)} ds \right].$$

The first term can be calculated using the fact that

$$\sigma \int_0^{T-t} \tilde{w}(s) ds = \sigma \int_0^{T-t} (T-t-u) d\tilde{w}(u)$$

is normally distributed with

$$\begin{aligned} \mathbf{E}^* \left[\sigma \int_0^{T-t} \tilde{w}(s) ds \right] &= 0 \\ \mathbf{Var}^* \left[\sigma \int_0^{T-t} \tilde{w}(s) ds \right] &= \frac{\sigma^2 (T-t)^3}{3} \end{aligned}$$

We obtain

$$X_t \mathbf{E}^* \left[e^{-\sigma \int_0^{T-t} \tilde{w}(s) ds} \right] = X_t e^{\frac{\sigma^2 (T-t)^3}{6}}$$

For the second term, we apply the Fubini's theorem to claim that

$$\begin{aligned} &\mathbf{E}^* \left[e^{-\sigma \int_0^{T-t} \tilde{w}(s) ds} \int_0^{T-t} \frac{P(0, s+t)}{P(0, s+t+\tau)} e^{\frac{\sigma^2}{2}(s+t+\tau)(s+t)\tau + \sigma\tau\tilde{w}(s)} ds \right] \\ &= \int_0^{T-t} \frac{P(0, s+t)}{P(0, s+t+\tau)} e^{\frac{\sigma^2}{2}(s+t+\tau)(s+t)\tau} \mathbf{E}^* \left[e^{-\sigma \int_0^{T-t} \tilde{w}(s) ds + \sigma\tau\tilde{w}(s)} \right] ds. \end{aligned}$$

Note that

$$-\sigma \int_0^{T-t} \tilde{w}(s) ds + \sigma\tau\tilde{w}(s) = -\sigma \int_0^{T-t} (T-t-u) d\tilde{w}(u) + \sigma\tau \int_0^s d\tilde{w}(u). \quad (18)$$

is normally distributed with

$$\begin{aligned} \mathbf{E}^* \left[-\sigma \int_0^{T-t} \tilde{w}(s) ds + \sigma\tau\tilde{w}(s) \right] &= 0 \\ \mathbf{Var}^* \left[-\sigma \int_0^{T-t} \tilde{w}(s) ds + \sigma\tau\tilde{w}(s) \right] &= \sigma^2 \left[\tau s^2 + \{\tau^2 - 2\tau(T-t)\}s + \frac{(T-t)^3}{3} \right] \end{aligned}$$

Therefore, the second term is

$$\exp \left[\frac{\sigma^2}{2}(t + \tau)t\tau + \frac{\sigma^2(T - t)^3}{6} \right] \int_0^{T-t} \frac{P(0, s + t)}{P(0, s + t + \tau)} \exp \left[\sigma^2\tau s^2 + \sigma^2(\tau^2 + 2t\tau - T\tau)s \right] ds$$

This leads to the desired expression for h . ■

When $X(t) < 0$, the price of the average rate call must be solved numerically, e.g., applying the finite difference method to find h at time t . The boundary condition at time T for h is

$$h(X, T) = \max[X, 0]$$

When X is large, the formula in the above proposition can be used as a boundary condition. And, when X is small, we note that

$$\lim_{X \rightarrow -\infty} h(X, t) = 0$$

Numerical examples using the finite difference method are shown in the next section.

As mentioned before, once the prices of European call options are obtained, the prices of European put options are easily derived through the put-call parity shown in the following proposition.

Proposition 4 *The put-call parity for the average rate options is given by*

$$P(t) = C(t) + P(t, T)K - \mathbf{E}_t^* \left[e^{-\int_t^T r(u)du} Z_T \right] \quad (19)$$

PROOF. The above relation is clearly true at the expiration date T . Discounting both sides by the money market rates and taking the conditional expectation, we can easily derive the above relation. ■

Note that

$$\begin{aligned} \mathbf{E}_t^* \left[e^{-\int_t^T r(u)du} Z_T \right] &= \frac{1}{T\tau} \mathbf{E}_t^* \left[e^{-\int_t^T r(u)du} \int_0^T \frac{1}{P(t, t + \tau)} dt \right] \\ &\quad - \frac{1}{\tau} \mathbf{E}_t^* \left[e^{-\int_t^T r(u)du} \right] \\ &= g(r, t)h(X, t) - \frac{1}{\tau} P(t, T) \end{aligned}$$

where g is defined by (12) and h is given by the explicit formula for $X_t \geq 0$ in proposition 2 with $k = 0$.

4 Numerical Examples

We now present two numerical examples which illustrate how our variable reduction technique can be efficiently used to value average-rate options. Our first example involves evaluating average-rate options written on foreign exchange rates, while our second example values average-rate options on one-year CMS rates.

Tables 1-3 show the prices of average-rate options (with continuous averaging) on dollar-yen exchange rates with three different expiration dates (i.e., three months, six months, and one year). For each expiration date, the prices of out-of-the money, at-the money, and in-the-the money options are shown separately. The spot price and the volatility are assumed to be 100 yen and 10 percent per year, respectively, while the risk-free interest rates for yen and dollar are assumed to be 3 percent and 5 percent, respectively. Option prices are quoted in terms of yen. In the first row of each table, we obtain the option prices by solving the PDE (3) using a Crank-Nicholson finite difference scheme, while in the second row of each table, we calculate the option prices by evaluating the expectation of (5), i.e., the Feynman-Kac representation of the solution of the PDE (3), using the standard Monte Carlo simulation applying to a discretized counterpart for X . Specifically, we discretize X using a standard first order finite difference scheme:

$$x_{n+1} = x_n + (1 - \alpha x_n)\Delta - \sigma x_n \sqrt{\Delta} \tilde{\epsilon}_n$$

A total of 100,000 trials are implemented in each simulation. For purposes of comparisons, we report in the last row of the table the option prices calculated using the more conventional Monte Carlo simulation method, i.e., simulate a sample path of the exchange rate process (i.e., S) and compute the average exchange rates (i.e., A) along each of the sample paths generated. A total of 500,000 paths have been sampled to arrive at the numbers reported. It is clear from these tables that in terms of accuracy, our variable reduction method fares well with Monte-Carlo simulations methods. Moreover, we note that among the three methods, the computing time involved in simulations is much longer than that of the finite difference method which generates the numbers in the first row (in order to achieve the same level of accuracy).⁶

Tables 4-6 show similar results for average-rate options written on 1-year constant maturity yields (CMS) with three different times to maturity, 3 months, 6 months and 1 year.

⁶The computing time required for implementing the finite difference scheme is well under one minute on a SunSparc 20 machine.

For each maturity, the prices of three different strikes are shown as in the case of the foreign exchange average-rate options. For simplicity, the term structure of interest rates is assumed to be flat at 5 percent in all cases, and the volatility of instantaneous forward rates is assumed to be 150 basis point per year. The option prices are expressed in terms of basis point per year. As in the case for foreign exchange rate options, the PDE for $h(X, t)$ is solved numerically by the Crank-Nicholson finite difference scheme. Option prices are also evaluated through Monte-Carlo simulations based on Feynman-Kac representation for h , where 100,000 trials are implemented for each case, and the more conventional simulations method, where 500,000 trials are implemented for each case. Once again, given the same level of accuracy, our variable reduction method takes much less time to compute.

5 Concluding Remarks

We have presented in this paper a variable reduction technique which values average-rate options by reducing the two-dimensional valuation problem to a one-dimensional problem. In doing so, it reduces significantly the computing time required for average-rate options. While we have shown that this technique is useful when the underlying state variable is lognormally distributed, this technique is also applicable when the returns of asset prices are independent of their past histories. An example of such is the 2-factor stochastic volatility model proposed by Hull and White (1987) in which the volatility of the underlying is governed by another one-dimensional Markov process. Applications of the variable reduction technique for other path-dependent derivative products (e.g., lookback options) are the subject for future research.

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Table 1: Average-Rate Options on FX (T=0.25y)

Strike price	105	100	95
Finite Difference	0.046	1.022	4.766
Monte Carlo (I)	0.046	1.024	4.765
Monte Carlo (II)	0.047	1.024	4.764

Table 2: Average-Rate Options on FX (T=0.50y)

Strike price	105	100	95
Finite Difference	0.183	1.366	4.679
Monte Carlo (I)	0.183	1.364	4.680
Monte Carlo (II)	0.184	1.368	4.679

Table 3: Average-Rate Options on FX(T=1.00y)

Strike price	105	100	95
Finite Difference	0.464	1.772	4.632
Monte Carlo (I)	0.465	1.765	4.616
Monte Carlo (II)	0.466	1.779	4.651

Table 4: Average-Rate Options on 1-Year CMS($T=0.25y$)

Strike rate %	5.50	5.00	4.50
Finite Difference	5.36	24.99	63.81
Monte Carlo (I)	5.34	25.11	63.68
Monte Carlo (II)	5.32	25.07	63.90

Table 5: Average-Rate Options on 1-Year CMS($T=0.50y$)

Strike rate %	6.00	5.00	4.00
Finite Difference	2.65	32.05	111.34
Monte Carlo (I)	2.64	31.99	111.00
Monte Carlo (II)	2.63	32.02	111.36

Table 6: Average-Rate Options on 1-Year CMS($T=1.00y$)

Strike rate %	6.00	5.00	4.00
Finite Difference	8.06	41.32	112.25
Monte Carlo (I)	8.03	41.36	112.42
Monte Carlo (II)	8.05	41.34	112.31